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AN
ELEMENTARY TREATISE
ON THE
DIFFERENTIAL CALCULUS,

IN WHICH THE METHOD OF LIMITS IS
EXCLUSIVELY MADE USE OF.

BY THE
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LATE FELLOW OF CAIUS COLLEGE, CAMBRIDGE.

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PREFACE.

THE method of Limits is generally allowed to be the best and most natural basis upon which to found the principles of the Differential Calculus; in the following pages this method is *exclusively* adopted, no use whatever being made of series in the demonstration of fundamental propositions. The following is an outline of the work, which is by no means offered to the reader as a complete treatise on the subject, but merely as an exposition of its more prominent and useful principles.

In Chap. I, certain terms, afterwards to be used, are defined and explained. In Chap. II the nature of a *Limiting Value* is fully set forth, and the important distinction (which ought never to be overlooked) between an *actual* and a *limiting* value is pointed out and illustrated by examples. Chap. III contains a set of *Lemmas*, which are necessary in order to render the use made of limiting values in the Differential Calculus perfectly legitimate; and here I have endeavoured to confine myself to what seems really essential. In Chap. IV certain important limiting values are obtained. Chap. V contains the *Rules for Differentiation*, in the demonstration of which Lagrange's *functional* notation is employed, as being the simplest to begin with. In Chap. VI the *Differential notation* of Leibnitz is explained, $\frac{dy}{dx}$ is defined as the *quote* of the *differentials* dy and dx, which however are not supposed to be infinitesimals, but simply two arbitrary quantities in a certain ratio. In the case of *partial* differential coefficients, some modification of the common differential notation $\frac{du}{dx}$, $\frac{du}{dy}$, is clearly necessary: I have employed the suffix notation d_xu, d_yu, as being frequently employed, though not exactly in this manner. I should have much preferred the notation d_xu, d_yu to denote partial *differentials*, and $\frac{d_xu}{dx}$ $\frac{d_yu}{dy}$ to denote partial *differential coefficients*. Chap. VII relates to *successive* differentiation, and the change of the *independant variable*. Chap. VIII contains certain very important *Lemmas* upon which the use and application of the Differential Calculus in a great measure depends. Chap. IX contains the theory of *Series*, based upon one of the preceding Lemmas, without assuming that $f(x+h)$ can be developed in the form

$$A + Bh^{\alpha} + Ch^{\beta} + \&c. \dots :$$

and here I have endeavoured to shew what the real nature of a series is, and to prove rigorously the principle of *Indeterminate Coefficients.* Chapters X and XI relate to *Vanishing Fractions, and Maxima and Minima,* and contain some useful simplifications of the common methods. The very insufficient and troublesome criterion usually employed in distinguishing the maxima and minima of functions of two variables is not introduced. Chap. XII relates to *Tangents, Normals,* &c., the *Curvature* of Curves, and the properties of the *Evolute;* and here the arrangement usually adopted is somewhat departed from, and what seems a more natural course pursued, in order to avoid certain difficulties, which I have observed very often impede the student on his first reading of the subject. In Chap. XIII the useful *Polar formulæ* and the differentials of *Areas, Volumes,* &c., are deduced. Chap. XIV relates to *Asymptotes.* Chap. XV contains a very simple method of *tracing curves.* Chap. XVI relates to *singular points.* Chap. XVII contains the general *Theory of Contacts* and *Ultimate Intersections:* no use is made of series in explaining the different orders of contact. The remaining chapters are occupied with *Elimination* by differentiation, *Lagrange's Theorem,* the properties of the *Cycloid,* &c., &c. The Appendix contains Examples worked out.

It was my intention to have added a few more chapters, and among the rest, one on the origin and progress of the Differential Calculus, and another on the Infinitesimal method; but from various circumstances I found it impossible to send the work to the press at the time originally promised to my bookseller, without omitting these concluding chapters. I mention this to account for the absence of allusions to the History of the Differential Calculus, which were all reserved for the final chapter, and the small number of Examples in the Appendix.

Professor Peacock's excellent collection of Examples, which have been of such service to the Mathematical Student, is now out of print; but Mr Gregory's work lately published will supply its place, which contains, not only a great number of well-selected and valuable examples, but also many important explanations and theorems not to be met with in any elementary treatise. In a subject of so much importance as the present, the student ought not to confine his attention to one book or system: for a very valuable treatise on this subject he is referred to that published by the Society for the Diffusion of Useful Knowledge.

In the general plan of this work, and in several particulars, I have deviated from some of the methods often made use of, partly in attempting to put the subject in a simpler and clearer point of view, and partly in avoiding certain steps of reasoning which appear to be defective. One of these is the fallacy of establishing premises on a certain implied condition, and drawing a conclusion from them by a direct violation of that condition. An example of this is to be found in a proof often given of the principle of indeterminate coefficients, in which the factor x is divided out of the equation

$$Bx + Cx^2 + Dx^3 + \&c. \ldots = 0,$$

which of course tacitly assumes the condition that x is not zero; in this manner is obtained the equation

$$B + Cx + Dx^2 + \&c. \ldots = 0;$$

and then by putting $x = 0$, contrary to the implied condition, the conclusion $B = 0$ is arrived at. Another example of this kind of reasoning is given in the first note, page 6.

The *assumption*, that $f(x+h)$ can be expanded in a series of the form $A + Bh^\alpha + Ch^\beta$ &c.... seems to me to be a serious defect in the common method of establishing Taylor's Series, and thereupon the principles of the Differential Calculus. This assumption is usually justified by arguing, that if we find definite values for A, B, C, &c. it shews that the assumption is correct. Now this argument may be stated thus:

"If the assumption that $f(x+h) = A + Bh^\alpha + Ch^\beta$ + &c. be true, then A, B, C, &c., must have definite values. But we can in general obtain definite values for A, B, C, &c.... (*e.g.* by the method of indeterminate coefficients.) Therefore the assumption is true."

This is clearly a falacious argument, for to warrant the conclusion the first premise should have been this:

"If the assumption *be not true*, definite values cannot be obtained for A, B, C, &c."

These defective steps of reasoning and others which might be mentioned, are objectionable, not because they lead to erroneous conclusions, but because they ought not to be found in a subject like the present, in which every thing should be comformable to the strictest rules of logical deduction. M. Cauchy has done much towards the improvement and perfection of the Differential Calculus, and his writings on this, like those on the more abstruse branches of mathematics, are most valuable. In one or two places the methods I have employed in the following pages are apparently similar to those of M. Cauchy, but in reality they are essentially different: so far as I am aware I am indebted to him only for article 48.

Cambridge,
October, 1842.

ERRATA.

The reader is requested to make the following correction which is of some importance.

3rd line from foot of page 75 instead of (2) *read* (0).

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THE

DIFFERENTIAL CALCULUS.

CHAPTER I.

PRELIMINARY REMARKS. VARIABLES AND CONSTANTS. FUNCTIONS, CONTINUOUS AND DISCONTINUOUS. ILLUSORY FUNCTIONS. EXAMPLES.

The nature of the Differential Calculus cannot be explained at first.

1. At the commencement of a treatise upon any science, the reader naturally expects to find some brief statement of the nature and object of that science; but in the present instance it is impossible to make such a statement, unless the meaning of certain terms, necessary to be used, be previously explained: for the Differential Calculus, from the very first, involves notions which must be new to one who is acquainted with no more than the common elements of Algebra and Analytic Geometry; and in the following pages we must suppose the reader to have advanced only so far in the study of mathematics.

We shall therefore make a few preliminary remarks, and explain certain fundamental notions, before we state what the Differential Calculus is.

Variables and Constants.

2. *Any quantity capable of variation is called a Variable.*

Any quantity which cannot vary, or, if it can, is supposed not to vary, is called a Constant.

Thus in the general equation to a right line, namely, $y = mx + c$, if we suppose the line never to change its position,

m and c are constants, and x and y are variables; but if we suppose that the line may change its position, m and c are variables as well as x and y, for they are then capable of variation.

It is usual to take the first letters of the alphabet a, b, c...&c. to denote constants, and the last letters...u, v, w, x, y, z to denote variables; but in many cases this distinction cannot be conveniently maintained, inasmuch as quantities which in one case we consider variable must often, in another, be considered constant, and vice versa.

Variables are often called, *Arbitrary quantities*, *Indeterminate quantities*, *Unassigned quantities*.

Continuous and Discontinuous Variables.

3. *A variable to which we may give in succession any number of different values, which differ from each other as little as we please, is called a Continuous Variable; otherwise it is said to be Discontinuous.*

Thus if x be taken to represent the length of any line which may be of any magnitude we please, then x is a continuous variable: but if x be taken to represent any *integer*, then it is a discontinuous variable; for in the former case we may give to x in succession any number of different values which differ from each other as little as we please; whereas in the latter case, though we may give x any number of different values, we cannot make them differ from each other as little as we please. The distinction between a continuous and a discontinuous variable may be briefly expressed by saying, that one admits of *gradual* change, but the other does not.

A Function defined. The Functional Notation explained.

4. *The result obtained by performing a certain operation or set of operations upon a variable* x, *is said to be a certain function of* x.

The words "*a certain function of*" are, for the sake of brevity, denoted by the letter f written before x. Thus the equation $y=f(x)$ simply means, that y is "*a certain function of*" x, i. e. that y is the result of performing a certain operation or set of operations upon x. The letter f, which we shall call the *Functional Letter*, is, as it were, the *represen-*

tative of the particular operation or set of operations performed upon x. Thus if $f(x) = x^n$, f represents the operation of taking a quantity to the n^{th} power; if $f(x) = \log \sin x$, f represents the operations of taking the logarithm of the sine of a quantity; if BPQ (fig. 1) be a given curve, $AM\ (=x)$ $MP\ (=y)$ the co-ordinates of any point P, and if we take $f(x)$ to represent y, then f represents the operations of measuring AM equal to x along the line AX, erecting MP perpendicular to AX, producing it to meet the curve at P, and so finding the length MP or y.

A function is not necessarily a quantity which changes when the variable changes.

5. From the definition here given of a function we may see that $f(x)$ is not necessarily a quantity which changes when x changes; for a set of operations performed upon x may sometimes lead to a result which is the same for all values of x. This will appear from the following example.

Let APB (fig. 2) be a semicircle whose radius is a, C its center; take any arc $AP = x$, with P as center, and some given line c as radius, describe a circle cutting AB at the point Q: then we may say that AQ is a function of x and denote it by $f(x)$; f will therefore represent the operations of measuring x along AP, describing a circle with radius c and center P, so finding Q, and therefore finally AQ.

Now in general AQ or $f(x)$ will be different for different values of x; but if we take $c = a$, then the circle described with P as center and $c\ (=a)$ as radius will always cut AB at the center C, and therefore we shall have $AQ = AC$ or $f(x) = a$, whatever be the value of x. Hence $f(x)$ in this case does not vary when x varies.

Thus it appears, according to our definition of a function, that $f(x)$ is not necessarily a quantity which changes when x changes; and this remark is important, as will appear hereafter.

6. When we have occasion to consider several different functions at the same time, we employ *different* functional letters, in order to distinguish between them. The letters commonly used, in addition to f, are F, ϕ, ψ, χ, and sometimes these letters with dashes, thus f', F', ϕ', ψ', χ' or f'', F'', &c...&c. Thus we might put $f(x)$ to represent x^n, $\phi(x)$

to represent log sin x, $\psi'(x)$ to represent y in the curve, fig. 1, &c.... &c.

Functions of several variables, as well as of one.

7. The result of performing a certain operation or set of operations upon *several* variables x, y, z, &c. is, in like manner, said to be a certain function of x, y, z, &c., and is represented similarly by the notation $f(x, y, z, ...)$. Thus $x^2 + y^2 + z^2$, $x \log (y + z)$, are the results of performing certain operations upon x, y, z, and are accordingly said to be certain functions of x, y, z, and we denote them by functional letters written before x, y, z, thus, viz.

$$x^2 + y^2 + z^2 = f(x, y, z) \qquad x \log (y + z) = \phi (x, y, z.)$$

Geometrical representation of a function.

8. We may evidently draw the curve BPQ (fig. 1) in such a manner that y shall be any function we please of x; and thus by means of a curve we may denote any function, and as it were represent it to the eye, which is often a very good method of illustrating general theorems respecting functions.

Functions of dependant and of independant variables.

9. In a function of several variables $f(x, y, z...)$ it may happen that the variables x, y, z... are connected with each other in some manner, so that we cannot change one without at the same time changing the others. Or it may happen that x, y, z...are not at all connected with each other, so that we may assign to each of them any value we please independently of the rest. In the former case $f(x, y, z...)$ is said to be a function of several *mutually dependant* variables, and in the latter case $f(x, y, z)$ is said to be a function of several *independant* variables.

Functions explicit and implicit.

10. A quantity y is said to be an *explicit* function of another, x, when we can state the precise operations by which y may be deduced from x; if not, y is said to be an implicit function of x. Thus if we are given the equation

$$y^5 - 3x^4y + x^5 = 0,$$

we know that there must be a certain set of operations by which y may be deduced from x, but what these operations are we cannot precisely state; in such a case y is called an implicit function of x.

The functions which commonly occur in mathematics are continuous.

11. The functions which commonly occur in mathematical investigations are of such a nature, that they always suffer a gradual and not a sudden change, so to speak, when the variable is gradually altered in value. Functions of this kind are called *Continuous Functions.* By saying that a function $f(x)$ always receives a gradual change when x is gradually varied, we mean this; that if x be changed into x', and therefore $f(x)$ into $f(x')$, then $f(x') \sim f(x)$ may be made as small as we please by taking $x' \sim x'$[*] small enough, whatever be the value of x. That this is true for all ordinary functions, such as x^n, a^x, $\log x$, $\sin x$, &c....&c. and all ordinary combinations of these functions, such as $(a^x + \log \sin x)^n$, $a^x \sin x$, &c...&c. does not require to be proved, for it is quite evident. *All ordinary functions therefore are continuous functions.*

Discontinuous functions.

12. Functions which sometimes suffer a sudden change when the variable is gradually altered in value, are called *Discontinuous Functions.* Thus, if we draw a broken curve as in (fig. 3) the ordinate will be a discontinuous function of the abscissa: for it is evident that the ordinate will suffer a sudden change at the points P, Q, R, S, supposing the abscissa to be gradually varied.

13. We shall never have occasion to consider discontinuous functions in the following pages, and therefore we shall always suppose that the functions we make use of are continuous.

Assumption which we shall always make respecting functions.

Hence, if $f(x)$ be any function of x we make use of, and if x be changed into x', and therefore $f(x)$ into $f(x')$; we shall always assume that $f(x') \sim f(x)$ may be made as small as we please by taking $x' \sim x$ small enough whatever be the value of x.

An illusory function defined.

14. *A function* f (x) *is said to become Illusory when the operations represented by* f *cease to give any definite result;* which may happen in certain cases, as we shall shew.

---

[*] $x' \sim x$ denotes the difference between x' and x, subtracting the greater of these quantities from the lesser; $x' \sim x$ is therefore the absolute difference between x' and x without regard to sign.

Ex. 1. A fraction which assumes the form $\frac{0}{0}$.

Thus, if $f(x) = \frac{x^2 - x}{x^2 - 1}$, the operations represented by f cease to give any definite result when $x = 1$; for then $f(x)$ assumes the form $\frac{1-1}{1-1}$, or $\frac{0}{0}$, which is not a definite result, as is shewn in the note*; $f(x)$ therefore becomes illusory when $x = 1$.

Ex. 2. $(1+x)^{\frac{1}{x}}$ when $x = 0$.

15. Again, if $f(x) = (1 + x)^{\frac{1}{x}}$, the operations represented by f cease to give a definite result when $x = 0$; for then $(1 + x)^{\frac{1}{x}}$ assumes the form $1^{\frac{1}{0}}$, which is not a definite result, as is shewn in the note†; $f(x)$ therefore becomes illusory, in this case, when $x = 0$.

Ex. 3. Case of two intersecting

16. Again, if *ABC* (fig. 4) be an ellipse marked with the usual letters, *PG* the line bisecting the angle *SPH*, and

---

$\frac{0}{0}$ does not represent any definite quantity.

* That $\frac{0}{0}$ is not a definite quantity appears thus. $\frac{0}{0}$ according to the strict definition of a quotient, is that quantity which multiplied by 0 gives 0: now any quantity whatever multiplied by 0 gives 0; therefore $\frac{0}{0}$ is any quantity whatever; i. e. it is not a definite quantity.

It may be said, however, that though $\frac{0}{0}$, considered absolutely, is not a definite quantity, nevertheless $\frac{x^2 - x}{x^2 - 1}$ becomes $\frac{1}{2}$ when $x = 1$; for $\frac{x^2 - x}{x^2 - 1} = \frac{x}{x+1}$, and $\frac{x}{x+1} = \frac{1}{2}$ when $x = 1$, and therefore $\frac{x^2 - x}{x^2 - 1} = \frac{1}{2}$ when $x = 1$.

To this we may answer, that $\frac{x^2 - x}{x^2 - 1}$ is proved to be equal to $\frac{x}{x+1}$ by dividing its numerator and denominator by $x - 1$; but we may not perform this division when $x - 1 = 0$, since there is no rule of Algebra which enables us to divide the numerator and denominator of a fraction by zero without altering its value; hence the equation $\frac{x^2 - x}{x^2 - 1} = \frac{x}{x+1}$ holds only on the express condition that x does not $= 1$; and therefore we may not draw any conclusion from this equation which requires us to suppose that x actually $= 1$: consequently, we cannot assert that $\frac{x^2 - x}{x^2 - 1} = \frac{1}{2}$ when $x = 1$ because $\frac{x^2 - x}{x^2 - 1} = \frac{x}{x+1}$.

$1^{\frac{1}{0}}$ not a definite quantity.

† That $1^{\frac{1}{0}}$ is not a definite quantity may be shewn thus. $1^{\frac{1}{0}}$ is that quantity which taken to the power 0 becomes 1; now any quantity whatever taken to the power 0 becomes 1; therefore $1^{\frac{1}{0}}$ is not a definite quantity.

$CM\,(=x)$ the abscissa of P; then if we assume $f(x) = CG$, the operations represented by f cease to give a definite result when $x = a$. For f represents the operations of drawing MP perpendicular to CM, so determining the point P, and the lines SP, HP, and then drawing PG to bisect $\angle SPH$, and so finding G, and therefore CG. Now when $x = a$, P coincides with A, and therefore the line PG with CA, and therefore PG cannot be said to intersect AC in one point more than another; therefore CG is not a definite quantity, and therefore the operations denoted by f cease to give a definite result when $x = a$. Hence $f(x)$ becomes illusory when $x = a$.

lines becoming coincident.

17. Again, if P (fig. 5) be a point on a curve PQ, RPQ a right line drawn through P and any other point Q on the curve, meeting the axis of x (AX) in the point R: then if we assume arc $PQ = s$, and $\angle PRX = \phi(s)$, the operations represented by ϕ cease to give a definite result when $s = 0$. For ϕ represents the operations of measuring $PQ = s$, drawing RPQ through P and Q, and so finding the angle PRX: now when $s = 0$, Q coincides with P, and any line whatever drawn through P passes also through Q; therefore the line QPR does not occupy a definite position, nor is $\angle PRX$ a definite angle, when Q coincides with P; and therefore the operations represented by f cease to give a definite result when $s = 0$. Hence $\phi(s)$ becomes illusory when $s = 0$.

Ex. 4. Case of a right line joining two points of a curve, when these points are made to coincide.

18. From these examples it is evident that a function may become illusory when x receives a particular value, the operations represented by the functional letter ceasing to give any definite result.

In each of these examples we may easily see that if x differ ever so little from that value which makes $f(x)$ illusory, the operations represented by f do lead to a definite result. It is therefore only for isolated values of x that $f(x)$ becomes illusory: and this will be found to be true in all cases where functions become illusory.

Functions become illusory only for isolated values of the variable.

CHAPTER II.

THE DISTINCTION BETWEEN AN ACTUAL AND A LIMITING VALUE EXPLAINED. A TANGENT DEFINED. THE NATURE OF THE DIFFERENTIAL CALCULUS STATED.

The distinction between an actual value and a limiting value.

19. THE Examples brought forward in the preceding chapter, for the purpose of shewing that a function may become illusory when the variable receives a particular value, lead us to make a very important though simple distinction, namely, the distinction between an *Actual Value* and a *Limiting Value* of a function.

An Actual Value of a function f(x) *is the result obtained by giving* x *some particular value, and performing upon it the operations represented by* f.

A Limiting Value of a function f(x) *is that quantity from which we may make* f(x) *differ as little as we please, by making* x *approach nearer and nearer in magnitude to some particular value without actually becoming equal to it.*

That the limiting value does not cease to be a definite quantity when the actual value does: shewn by examples.

20. An actual and a limiting value thus defined seem at first sight to be the same thing; and indeed, as long as the operations represented by f lead to a definite result, they are identical, as we shall prove presently; it is only when the function becomes illusory that the distinction between them is real, and it consists in this, that *the actual value ceases to be a definite quantity, whereas the limiting value does not.* This we shall shew in the case of the examples just alluded to.

Ex. 1. A fraction which assumes the form $\frac{0}{0}$.

21. Let us consider the first example, namely

$$f(x) = \frac{x^2 - x}{x^2 - 1}, \quad \text{(see 14)}.$$

We have seen that the operations here denoted by f give no definite result when $x = 1$, or, in other words, that the actual value ceases to be a definite quantity when $x = 1$.

But not so the limiting value; for we may divide the numerator and denominator of $f(x)$ by $x - 1$, and so arrive at the equation $f(x) = \frac{x}{x+1}$, except when x actually $= 1$; now by making x approach nearer and nearer in magnitude to 1, without actually becoming equal to it, we may evidently make $\frac{x}{x+1}$ differ as little as we please from $\frac{1}{2}$, and therefore the same may be said of $f(x)$, since it ceases to be equivalent to $\frac{x}{x+1}$ only when x actually $= 1$: $\frac{1}{2}$ is therefore a quantity from which we may make $f(x)$ differ as little as we please, by making x approach nearer and nearer in magnitude to 1, without actually becoming equal to it.

And it is easy to see that there is no other quantity but $\frac{1}{2}$ from which we may make $\frac{x}{x+1}$, and therefore $f(x)$, differ as little as we please by making x approach nearer and nearer to 1. Hence it appears that $\frac{1}{2}$, and no other quantity but $\frac{1}{2}$, is the limiting value of $f(x)$ when x approaches 1. We see therefore in this case that when the actual value ceases to be a definite quantity the limiting value does not.

Ex. 2. Two intersecting lines which become coincident.

22. We are not yet sufficiently advanced to shew that the same is true in the case of the second example (15); so we shall pass on to the third example (16).

We have seen that the operations here represented by f cease to give a definite result when $x = a$, i. e. the actual value of $f(x)$ ceases to be a definite quantity when $x = a$. But not so the limiting value; for since PG bisects $\angle HPS$ we have

$$\frac{HG}{SG} = \frac{HP}{SP}, \quad \text{or} \quad \frac{ae + CG}{ae - CG} = \frac{a + ex}{a - ex} \quad \text{(by Conic Sections)};$$

$$\text{and } \therefore\ CG \text{ or } f(x) = e^2 x\,^*, \text{ except } x \text{ actually } = a.$$

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[*] This proof evidently fails when $x = a$, for then P coincides with A and the triangle HPS ceases to exist.

Now e^2x may be made to differ from e^2a as little as we please by making x approach nearer and nearer in magnitude to a; therefore, as in the first example, e^2a, and no other quantity but e^2a, is the limiting value of $f(x)$ when x approaches a. Hence in this case when the actual value ceases to be a definite quantity the limiting value does not.

Ex. 3. A line cutting a curve in two points which become coincident.

23. Lastly, let us consider the fourth example (see 17).

We have seen that the operations here represented by ϕ cease to give a definite result when $s = 0$; i. e. the actual value of $\phi(s)$ ceases to be a definite quantity when $s = 0$. But not so the limiting value; for let $AM\,(=x)$ $MP\,(=y)$ be the co-ordinates of the point P (fig. 6), $AN\,(=x')$ $NQ\,(=y')$ the co-ordinates of the point Q, draw OP parallel to MN, and let $y = f(x)$ be the equation to the curve: then we evidently have

$$\tan PRX = \frac{OQ}{OP} = \frac{y' - y}{x' - x} = \frac{f(x') - f(x)}{x' - x},$$

and this equation is true, no matter how near x' may be to x, provided x' be not actually equal to x, for then the triangle OPQ ceases to exist, and $\tan PRX$ and $\frac{f(x') - f(x)}{x' - x}$ cease to have definite values.

Now for simplicity let us suppose the curve to be a parabola having the axis of y for its axis, and its equation being accordingly $y = \frac{x^2}{4m}$; then

$$\frac{f(x') - f(x)}{x' - x} = \frac{1}{4m}\,\frac{x'^2 - x^2}{x' - x} = \frac{x' + x}{4m},$$

except when $x' = x$.

Hence it is evident that, except when x' actually $= x$, we have

$$\phi(s) = \angle PRX = \tan^{-1}\frac{x' + x}{4m}.$$

Now the second member of this equation may be made to differ from $\tan^{-1}\dfrac{x}{2m}$ as little as we please by making x' approach x, or, what is the same thing, by making s approach zero; hence (as in the first example) $\tan^{-1}\dfrac{x}{2m}$, and no other quantity, is the limiting value of $f(s)$ when s approaches zero.

In a similar manner we might shew, in the case of other curves, that $f(s)$ has a definite limiting value when s approaches zero.

It appears therefore in this case, that when the actual value ceases to be a definite quantity the limiting value does not.

What a tangent to a curve is.

24. This last example leads us to the best and most accurate conception of what a tangent is. For draw the line SPT making the angle $\tan^{-1}\dfrac{x}{2m}$ with the axis of x; then by what has been proved $\angle PTX$ is the limiting value of $\angle PRX$ when Q approaches P; i.e. $\angle PRX$ may be made to differ from $\angle PTX$ as little as we please by making Q approach P without actually coming up to it; or what is the same thing, $\angle QPS$ may be made as small as we please by making Q approach P without actually coming up to it.

Now this being the case, it is natural to say that the line SPT just *touches* the curve at the point P, or that it is the *tangent* to the curve at P. Hence we define a tangent in the following manner.

Definition of a tangent.

25. If SPT be that line to which the secant RPQ may be made to approach nearer and nearer so as to make with it an angle as small as we please, by making Q approach P without actually coming up to it; then SPT is said to be the line touching the curve at the point P, or the tangent at P. Or to speak more briefly; *If* SPT *be the limiting position of the secant* RPQ *when* Q *approaches* P, *it is said to be the tangent at* P*.

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* This definition of a tangent seems to me to be the only accurate one that can be given, so as to apply to all cases of contact, such as contact at a point of con-

What the Differential Calculus is briefly stated.

26. The example we have just been considering, will enable us now to state generally the nature and object of the Differential Calculus.

We have seen, that to determine the position of the tangent at any point of a curve, as above defined, we have only to find the limiting value of $\frac{f(x') - f(x)}{x' - x}$ when x' approaches x; it appears therefore that we may arrive at a general method of drawing tangents to curves by means of this limiting value, if we can find it in all cases.

But this is a very small part of the use which may be made of this important limiting value: for there are very few branches of exact science which are not largely indebted to its assistance for the progress they have made. Indeed, without it, some of the most interesting applications of mathematics to the explanation of natural phænomena could never have been effected.

Now the Differential Calculus is that branch of mathematics whose object is, in the first place, to determine a set of rules whereby the limiting value of $\frac{f(x') - f(x)}{x' - x}$ when x' approaches x may be found with facility in all cases; and, in the second place, to explain some of the principal uses which may be made of this limiting value in pure mathematics.

The origin of the name *Differential Calculus* we shall presently explain.

trary flexure, contact at a cusp. It certainly is not correct to define the tangent STP to be the position which the line QPR assumes when Q coincides with P, for we have seen that the line QPR has no definite position when Q coincides with P.

CHAPTER III.

CERTAIN LEMMAS RESPECTING LIMITING VALUES.

27. In the preceding Chapter it was our object to shew that there is a real distinction between an actual and a limiting value in certain cases, and to state briefly the nature of the Differential Calculus. We now proceed to prove certain Lemmas, and to obtain certain limiting values which we shall find useful hereafter. But we must make a few remarks previously.

The phrase "diminish ad libitum," explained.

28. When a quantity may be continually diminished, so as to become less than any specified quantity, however small, without becoming actually zero, we shall say that it may be "*diminished ad libitum.*" We shall find this phrase convenient and perhaps less likely to be misunderstood than the words "*diminish indefinitely,*" which are generally used in the same sense.

29. Employing this phrase, we may state the assumption made in (13) as follows: viz.

Assumption made in Art. 13, stated somewhat differently.

By continually diminishing $x' \sim x$, *when it has once become sufficiently small, we may diminish* $f(x') \sim f(x)$ ad libitum; $f(x)$ *being any function, and* x *any value of the variable. And more generally by continually diminishing* $x' \sim x$, $y' \sim y$, $z' \sim z$ ... *when they have once become sufficiently small, we may diminish* $f(x'y'z'...) \sim f(xyz...)$ ad libitum; $f(xyz...)$ *denoting any function of several variables, and* $x, y, z...$ *any values of these variables.*

Of course we here suppose that $f(x)$ $\{$or $f(x, y, z...)$ if there be more than one variable$\}$ is not illusory.

30. And the definition of a limiting value in (19) may be stated thus:

Definition of a limiting value stated somewhat differently.

If by continually diminishing $\mathrm{x} \sim \mathrm{a}$, *when it has once become sufficiently small, we may diminish* $\mathrm{f(x)} \sim \mathrm{A}$ ad libitum; *then* A *is said to be the limiting value of* f(x), *when* x *approaches* a.

It is clear that the assumption and definition thus stated are equivalent to what they were before, only that they are more exactly expressed and better adapted to the use we shall have to make of them hereafter.

$f(x) \sim A$ does not in general diminish with $x \sim a$ until $x \sim a$ has become less than a certain value.

31. It is important to notice, that in this definition of a limiting value we do not assert that $f(x) \sim A$ must diminish when $x \sim a$ is diminished, for all values of $x \sim a$, but only when $x \sim a$ has once become sufficiently small. There are many cases in which $f(x) \sim A$ increases as $x \sim a$ diminishes, and continues to increase until $x \sim a$ has become less than a certain value, after which it continually diminishes with $x \sim a$.

To avoid circumlocution, instead of saying, "By diminishing $x \sim a$, when once it has become sufficiently small, we diminish $f(x) \sim A$ *ad libitum*," we shall simply say, "By *sufficiently* diminishing $x \sim a$ we diminish $f(x) \sim A$ *ad libitum*."

x may be a discontinuous variable, provided it be possible to diminish $x \sim a$ ad libitum.

32. It is also important to observe, that it is not essential to our conception of a limiting value, as above defined, that x should be a continuous variable (see 3); all that is necessary is this, that it be possible to diminish $x \sim a$ *ad libitum*, that is, to make it less than any specified quantity however small, without actually making it zero.

Thus if we suppose $x = a + \frac{1}{n}$, where n is always an integer, and therefore x not a continuous variable, we may conceive the existence of a limiting value of $f(x)$ when x approaches a (which it does when n approaches infinity), just as well as if x varied continuously: for by increasing n, though it be an integer, we may diminish $\frac{1}{n}$ or $x \sim a$, *ad libitum;*

which is all that is supposed necessary so far as $x \sim a$ is concerned in our definition of a limiting value.

But if we suppose $x = a + n$, where n is always an integer, then it is impossible to conceive the existence of a limiting value of $f(x)$ when x approaches a, since we cannot make $x \sim a$ or n less than unity unless we make it actually zero.

We now proceed, as we stated, to prove certain Lemmas, which will enable us to reason more exactly hereafter, and will now serve to illustrate the nature of limiting values.

33. *The limiting value of* f(x) *when* x *approaches* a *is the same whatever sort of variable* x *be, provided of course* $x \sim a$ *may be diminished* ad libitum. Lemma I.

Let x and z be two different variables such that both $x \sim a$ and $z \sim a$ may be diminished *ad libitum*, and let A be the limiting value of $f(x)$ when x approaches a. Then by *sufficiently* diminishing $x \sim a$ and $z \sim a$, and therefore $z \sim x$, we diminish *ad libitum*, $f(x) \sim A$ by the definition in (30), $f(x) \sim f(z)$ by the assumption in (29), and therefore $f(z) \sim A$. Hence by *sufficiently* diminishing $z \sim a$ we diminish $f(z) \sim A$ *ad libitum*, and therefore, by the definition (30), A is the limiting value of $f(x)$ when z approaches a, as well as that of $f(x)$ when x approaches a: from which the truth of the Lemma is manifest.

Thus if x be a continuous variable, and z a discontinuous variable in the form $a + \frac{1}{n}$, the limiting value of $f(x)$ when x approaches a, is also that of $f(z)$ when z approaches a. Examples.

Again, the limiting value of $f(x)$ when x approaches a is just the same thing as that of $f(e^x)$ or of $f(\tan x)$ when e^x or $\tan x$ respectively approaches a.

34. *If* f(x), *when expressed in terms of another variable* z, *becomes* ϕ(z), *and if* x = a *when* z = b; *then the limiting value of* f(x) *when* x *approaches* a, *and the limiting value of* ϕ(z) *when* z *approaches* b, *are the same thing.* Lemma II.

For let A be the former limiting value; then, since $x = a$

when $z = b$, it is evident that by *sufficiently* diminishing $z \sim b$ we may diminish $x \sim a$ *ad libitum*, and therefore $f(x) \sim A$, by definition (30), and therefore $\phi(z) - A$ since $\phi(z) = f(x)$; therefore A is the limiting value of $\phi(z)$ when z approaches b, by def. (30). Q. E. D.

Examples. Thus if in the function $f(x)$ we put $x = a + \frac{1}{z}$, and therefore $f(x) = f\left(a + \frac{1}{z}\right) = \phi(z)$ suppose; then $x = a$ when $z = \infty$, and therefore the limiting value of $f(x)$ when x approaches a is the same thing as that of $\phi(z)$ or $f\left(a + \frac{1}{z}\right)$ when z approaches ∞.

Again, if $f(x) = \frac{\epsilon^x - 1}{x}$, and we put $\epsilon^x = z$, and therefore $x = \log z$, and therefore $f(x) = \frac{z - 1}{\log z} = \phi(z)$ suppose; then x evidently $= 0$ when $z = 1$, and therefore the limiting value of $f(x)$ or $\frac{\epsilon^x - 1}{x}$ when x approaches 0, and that of $\phi(z)$ or $\frac{z - 1}{\log x}$ when z approaches 1, are the same thing.

We shall find this method of transforming expressions very useful in finding their limiting values.

Lemma III. 35. *Every Actual value is also a Limiting value.*

We have defined $f(a)$ to be the result of performing the operations represented by f upon a; therefore, by the definition of an actual value in (19), $f(a)$ represents the actual of $f(x)$ when $x = a$. But, from the assumption stated in (29), and the definition of a limit in (30), it is evident that $f(a)$ is the limiting value of $f(x)$ when x approaches a. Hence the truth of the lemma is evident. Of course we here suppose that $f(a)$ is not illusory.

Lemma IV. 36. *If* A *be the Limiting value of* f (x) *when* x *ap-*

proaches a, f (x) *has the same sign as* A *for all values of* x *taken sufficiently near* a.

Since $f(x) \sim A$ may be diminished *ad libitum* by taking x near enough to a, it is clear that $f(x)$ may be made greater or less than zero, according as A is greater or less than zero; or in other words, $f(x)$ may be made to have the same sign as A by taking x near enough to a: and when x is made to approach still nearer to a, since we so diminish $f(x) \sim A$ still more, at least for all values of x near enough to a (see 31), $f(x)$ must *continue* to have the same sign as A. Hence $f(x)$ has the same sign as A for all values of x taken near enough to a. Q. E. D.

Lemma V.

37. *If* f (x) *be any function of* x *which becomes illusory when* x = *a certain value* a, *and if for each value of* x (a *of course excepted*) f (x) *has only one value; then* f (x) *cannot have more than one limiting value when* x *approaches* a.

If possible let two different quantities A and B be both limiting values of $f(x)$ when x approaches a; then we may make $f(x)$ differ from both A and B as little as we please, at the same time, by *sufficiently* diminishing $x \sim a$; therefore x may be so taken, that A and B shall differ from the same quantity $f(x)$ {since $f(x)$ has only one value}, and therefore from each other, as little as we please; which is absurd if A and B be two different quantities; therefore A must be equal to B. Hence there cannot be more than one limiting value. Q. E. D.

38. Hence if we can prove that A is a limiting value of $f(x)$ when x approaches a, we are sure that no other quantity but A is a limiting value, and therefore that A is *the* limiting value.

A limiting value is not a mere approximate quantity.

Hence it appears that a limiting value is not a mere approximation, but a perfectly definite quantity; for if it were a mere approximation, then, when we had found a limiting value A, any quantity differing very little from A would be just as much a limiting value as A; contrary to what has been just proved.

We have supposed that the function $f(x)$ has only one value for each value of x; if however it has more than one, n values suppose, it is evident that there will be n different limiting values, and no more, when x approaches a. Thus if

$$f(x) = a + b\left(2\frac{x^2 - x}{x^2 - 1}\right)^{\frac{1}{2}}$$

which has two values for each value of x, then there are two limiting values when x approaches 1; viz. $a + b$ and $a - b$.

Lemma VI. 39. *If* $\mathrm{f}(\mathrm{x})$ *and* $\phi(\mathrm{x})$ *be two functions, one of which,* $\mathrm{f}(\mathrm{x})$, *becomes illusory when* $\mathrm{x} = $ *a certain value* a, *and the other,* $\phi(\mathrm{x})$, *does not; and if we can shew that* $\mathrm{f}(\mathrm{x}) = \phi(\mathrm{x})$ *for all values of* x, a *of course excepted; then* $\phi(\mathrm{a})$ *is the limiting value of* $\mathrm{f}(\mathrm{x})$ *when* x *approaches* a.

For by (29), we diminish $\phi(x) \sim \phi(a)$ *ad libitum* by *sufficiently* diminishing $x \sim a$; but in so doing we never suppose x to become actually equal to a, and we are therefore sure that $f(x) = \phi(x)$; therefore, we diminish $f(x) \sim \phi(a)$ *ad libitum*, by *sufficiently* diminishing $x \sim a$; i. e. $\phi(a)$ is the limiting value of $f(x)$ when x approaches a. Q.E.D.

Cor. 1. If $\phi(a)$ be illusory as well as $f(a)$, and if we know A to be the limiting value of $\phi(x)$ when x approaches a; then we may shew, in exactly the same way, that A is also the limiting value of $f(x)$ when x approaches a.

Cor. 2. If, instead of being able to shew that $f(x) = \phi(x)$ for all values of x except a, we can prove that $f(x) \sim \phi(x)$ is diminished *ad libitum* by *sufficiently* diminishing $x \sim a$; then the same conclusions evidently follow; that is to say; the limiting value of $f(x)$ when x approaches a is $\phi(a)$, or A if $\phi(a)$ be illusory.

Lemma VII. 40. *If* $\mathrm{f}(\mathrm{x})$ *be a function which becomes illusory when* $\mathrm{x} = \mathrm{a}$, *and if we can prove that* $\mathrm{f}(\mathrm{x})$ *lies between* another function* $\phi(\mathrm{x})$ *and a constant* A *for all values of* x *taken sufficiently near* a; *and moreover, that* A *is the limiting*

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* When we say that $f(x)$ lies between $\phi(x)$ and A, we mean that $f(x)$ is not greater than one of these quantities and not less than the other.

value of $\phi(\mathrm{x})$ *when* x *approaches* a*; then* A *is also the limiting value of* $\mathrm{f}(\mathrm{x})$ *when* x *approaches* a.

For since A is the limiting value of $\phi(x)$ when x approaches a, $\phi(x) \sim A$ is diminished *ad libitum* when $x \sim a$ is *sufficiently* diminished; therefore, *à fortiori*, since $f(x)$ lies between $\phi(x)$ and A, $f(x) \sim A$ is also diminished *ad libitum* at the same time; therefore A is the limiting value of $f(x)$ when x approaches a. Q.E.D.

Lemma VIII.

41. *If* U, V, W... *be any functions of* x, *and* $\mathrm{f}(\mathrm{U}, \mathrm{V}, \mathrm{W}\ldots)$ *any function of these functions, and if* A, B, C... *be the limiting values of* U, V, W... *respectively when* x *approaches a certain value* a*; then* $\mathrm{f}(\mathrm{A}, \mathrm{B}, \mathrm{C}\ldots)$ *is the limiting value of* $\mathrm{f}(\mathrm{U}, \mathrm{V}, \mathrm{W}\ldots)$ *when* x *approaches* a.

For by *sufficiently* diminishing $x \sim a$ we diminish $U \sim A$, $V - B$, $W - C\ldots$ *ad libitum*, and therefore, by the assumption in (29), we diminish $f(U, V, W\ldots) \sim f(A, B, C\ldots)$ *ad libitum;* therefore $f(A, B, C\ldots)$ is the limiting value of $f(U, V, W\ldots)$ when x approaches a.

Examples.

Thus the limiting value, when x approaches a,

of $U \pm V$ is $A \pm B$,

that of UV is AB,

that of $\dfrac{U}{V}$ is $\dfrac{A}{B}$,

that of $(U^2 + V^2) \sin W$ is $(A^2 + B^2) \sin C$.

In the proof of this Lemma, since we only speak of the *limiting* values of $W, V, W\ldots$ we make no supposition as to whether $U, V, W\ldots$ become illusory or not when $x = a$; so that the Lemma is equally true whether they do or whether they do not. If any of these quantities, U suppose, does not become illusory when $x = a$, then A is of course its actual value.

This Lemma evidently fails when the substitution of one or more of the quantities $A, B, C\ldots$ for $U, V, W\ldots$ respectively makes f illusory.

Example of the use of Lemma VIII. combined with some of the preceding Lemmas.

42. The following example will shew the use of Lemma VIII. combined with some of the preceding Lemmas.

$$\text{Let}\quad U = \frac{x^2 - 3x + 2}{x^2 - 1}, \quad \text{and}\quad V = \frac{2^{2x} - 2^{x+1}}{2^{2x} + 2^x - 6},$$

and suppose we wish to find the limiting value of

$$\frac{U^2 + UV + 10V^2}{10V - U} \sin^{-1}\{(U + 7)\, V\},$$

when x approaches 1.

Then, dividing out $x - 1$, $U = \dfrac{x - 2}{x + 1}$ except when $x = 1$, and $\dfrac{x - 2}{x + 2} = -2$ when $x = 1$. Therefore by Lemma VI. -2 is the limiting value of U when x approaches 1.

Again putting $2^x = z$, V becomes $\dfrac{z^2 - 2z}{z^2 + z - 6}$, and $x = 1$ when $z = 2$; hence by Lemma II. the limiting value of V when x approaches 1, is the same thing as that of $\dfrac{z^2 - 2z}{z^2 + z - 6}$, when z approaches 2. Now except when $z = 2$,

$$\frac{z^2 - 2z}{z^2 + z - 6} = \frac{z}{z + 3} \text{ which } = \tfrac{1}{5} \text{ when } z = 2;$$

therefore by Lemma VI. $\frac{1}{5}$ is the limiting value of V when x approaches 1. Hence by Lemma VIII. the limiting value of

$$\frac{U^2 + UV + V^2}{10V - U} \sin^{-1}\{(U + 7)\, V\},$$

when x approaches 1 is

$$\frac{4 - \frac{2}{5} + \frac{10}{25}}{\frac{10}{5} + 2} \sin^{-1}\{(-2 + 7)\tfrac{1}{5}\} = \frac{5\pi}{2}.$$

43. In proving the above Lemmas we have supposed, that functions become illusory only for isolated values of the variable; or in other words, that, if $f(a)$ be illusory, $f(x)$

is not illusory for all the values of x extremely near a, but only for the single value a. This assumption is evidently true for all ordinary functions.

We have not assumed the existence of a limiting value when the function becomes illusory in any of these Lemmas, inasmuch as they are worded in this manner; "*If* A be the limiting value then such and such things follow": nor shall we have any occasion to make this assumption; for in all cases where we have to consider the limiting values of functions when they become illusory, the existence of such values will be proved and not assumed. We may just observe however, that the existence of limiting values of functions when they become illusory is a necessary consequence of their becoming illusory only for isolated values of the variable, and of the assumption in (29), as it is not difficult to see.

CHAPTER IV.

CERTAIN LIMITING VALUES DETERMINED WHICH WE SHALL REQUIRE TO KNOW HEREAFTER.

Lemma IX. 44. If s *be the length of any arc of a curve, and* c *the length of its chord; the limiting value of* $\frac{s}{c}$ *when* s *approaches zero is unity.*

Let PSQ be the arc (fig. 7), PCQ the chord, draw PR the tangent at P, and QR perpendicular to PR, and let $\angle RPQ = \omega$: then, as long as P does not actually coincide with Q, we may evidently assume that PSQ lies between PCQ and $PR + QR$ (supposing of course that Q is taken near enough to P, so that the curve shall always bend towards the same side between P and Q); i.e. we may assume that

$$s \text{ lies between } c \text{ and } c\cos\omega + c\sin\omega;$$

$$\text{and therefore that } \frac{s}{c} \text{ lies between } 1 \text{ and } \cos\omega + \sin\omega.$$

Now, by the definition of a tangent in (25), 0 is the limiting value of ω, and therefore 1 that of $\cos\omega + \sin\omega$, when s approaches 0: therefore by Lemma VII, 1 is the limiting value of $\frac{s}{c}$ when s approaches 0. Q.E.D.

Lemma X. 45. *If* θ *be an angle measured by the subtending arc of a circle, the limiting value of* $\frac{\theta}{\sin\theta}$, *and that of* $\frac{\theta}{\tan\theta}$, *when* θ *approaches zero, are each unity.*

We may evidently assume, for all values of θ less than $\frac{\pi}{2}$, that

$$\theta \text{ lies between } \sin\theta \text{ and } \tan\theta,$$

and therefore that $\frac{\theta}{\sin\theta}$ lies between 1 and $\frac{1}{\cos\theta}$.

Now $\frac{1}{\cos\theta} = 1$ when $\theta = 0$; hence by Lemma VII. 1 is the limiting value of $\frac{\theta}{\sin\theta}$ when θ approaches 0. Q. E. D.

46. We might have deduced this result from the previous Lemma, by putting $s = 2\theta$, and c (which now becomes the trigonometrical chord of s) $= 2\sin\theta$, and therefore

$$\frac{s}{c} = \frac{\theta}{\sin\theta}.$$

Lemma XI. 47. *To find the limiting value of* $\frac{x^n - 1}{x - 1}$ *when* x *approaches unity,* n *being any number positive or negative integral or fractional.*

It is clear that, whether n be positive or negative fractional or integral, it may always be expressed in the form $\frac{p-q}{r}$, where p, q, and r are positive integers. Hence we may put

$$\frac{x^n - 1}{x - 1} = \frac{x^{\frac{p-q}{r}} - 1}{x - 1} = \frac{z^{p-q} - 1}{z^r - 1} \text{ if we put } x = z^r$$

$$= \frac{1}{z^q} \frac{(z^p - 1) - (z^q - 1)}{z^r - 1}.$$

Hence, dividing $(z^p - 1)$ $(z^q - 1)$ and $(z^r - 1)$ by $z - 1$, observing that p, q, r are positive integers, we have

$$\frac{x^n - 1}{x - 1} = \frac{1}{z^q} \cdot \frac{(1 + z + z^2 \ldots\ldots p \text{ terms}) + (1 + z + z^2 \ldots\ldots q \text{ terms})}{1 + z + z^2 \ldots\ldots r \text{ terms}}$$

except of course when $z = 1$.

Now when $x = 1$, $z = 1$, and therefore the actual value of the second member of this equation is evidently $\frac{p-q}{r}$ or n: hence, by Lemma VI, n is the limiting value of $\frac{x^n - 1}{x - 1}$ when x approaches 1.

Lemma XII. 48. *To shew that there exists a limiting value of* $\left(1 + \frac{1}{\mathrm{n}}\right)^{\mathrm{n}}$ *when* x *approaches infinity,* n *being always an integer.*

By the Binomial Theorem for positive integers we have

$$\left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{1}\,\frac{1}{n} + \frac{n.n-1}{1.2}\,\frac{1}{n^2} \text{......to } (n+1) \text{ terms}$$

$$= 1 + 1 + \frac{1}{\Gamma 2}\left(1 - \frac{1}{n}\right) + \frac{1}{\Gamma 3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \text{.........} (1),$$

where $\Gamma 2$, $\Gamma 3$, ... and in general Γr, denote (1.2), $(1.2.3)$, ... $(1.2.3...r)$, respectively.

Now in the series (1) all the terms are evidently positive, and when n increases each term increases, and the number of terms also increases. Moreover, this series is term by term less than the series

$$1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \text{.... to } n + 1 \text{ terms,}$$

which is $< 1 + \frac{1}{1 - \frac{1}{2}}$ * or 3.

Hence $\left(1 + \frac{1}{n}\right)^n$, which $= 2$ when $n = 1$, continually increases as n increases, but never comes up to 3, no matter how large n becomes. Therefore it is clear that there must be some number between 2 and 3 to which we may make $\left(1+\frac{1}{n}\right)^n$ *approach *ad libitum* by making n approach ∞, which number is the limiting value whose existence we wish to prove.

---

* A geometric series $1 + z + z^2$......to p terms $= \frac{1 - z^p}{1 - z} = \frac{1}{1 - z} - \frac{z^p}{1 - z}$. Now if z be positive and < 1, $\frac{z^p}{1 - z}$ is a positive quantity, and therefore the series $= \frac{1}{1 - z} -$ a positive quantity; therefore it is $< \frac{1}{1 - z}$, and this is true no matter how large p may be. *Hence, if* z *be a positive quantity less than unity,* $1 + z + z^2 + z^3$......... *continued to any number of terms is* $< \frac{1}{1 - z}$.

49. *To determine numerically the limiting value of* $(1+n)^{\frac{1}{n}}$ *when* n *approaches* ∞. Lemma XIII.

The $(r+1)^{\text{th}}$ term of the series (1) in the last article is

$$\frac{1}{\Gamma r}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)(\ldots\ldots)\left(1-\frac{r-1}{n}\right),$$

which is clearly $< \frac{1}{\Gamma r}$.

Hence the $(r+1)^{\text{th}}$ and following terms of the series (1) form a positive quantity less than

$$\frac{1}{\Gamma r}+\frac{1}{\Gamma(r+1)}+\frac{1}{\Gamma(r+2)}\ldots\ldots$$

which is $< \frac{1}{\Gamma r}\left(1+\frac{1}{r}+\frac{1}{r^2}\ldots\ldots\right)$,

which (by note, 48) is

$$< \frac{1}{\Gamma r}\left(\frac{1}{1-\frac{1}{r}}\right), \quad \text{or} \quad \frac{1}{\Gamma(r-1)}\cdot\frac{1}{r-1},$$

which is $< \frac{1}{\Gamma(r-1)}$.

Hence $\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{\Gamma 2}\left(1-\frac{1}{n}\right)$

$$+\frac{1}{\Gamma 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots \text{ to } r \text{ terms}$$

$$+ \text{ a positive quantity less than } \frac{1}{\Gamma(r-1)}.$$

---

* By saying that one quantity u *approaches* another v *ad libitum*, we simply mean that $u \sim v$ diminishes *ad libitum*.

If therefore we suppose $r = 12$, and consider only the first seven decimal places, it is clear from the table in the note*, that

$$\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{\Gamma 2}\left(1-\frac{1}{n}\right)$$

$$+ \frac{1}{\Gamma 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \text{...... to 12 terms;}$$

and this is true, no matter how large n may be.

Now by making n approach ∞, the second term of this equation may, term by term, be made to approach *ad libitum* to the series

$$1 + 1 + \frac{1}{\Gamma 2} + \frac{1}{\Gamma 3} \text{...... to 12 terms;}$$

which to 7 decimal places $= 2.7182818$ (by table, *).

---

* $1 = 1.00000000$......

$\frac{1}{\Gamma 2} = .50000000$......

$\frac{1}{\Gamma 3} = .16666666$......

$\frac{1}{\Gamma 4} = .04166666$......

$\frac{1}{\Gamma 5} = .00833333$......

$\frac{1}{\Gamma 6} = .00138888$......

$\frac{1}{\Gamma 7} = .00019841$......

$\frac{1}{\Gamma 8} = .00002480$......

$\frac{1}{\Gamma 9} = .00000275$......

$\frac{1}{\Gamma 10} = .00000027$......

$\frac{1}{\Gamma 11} = .00000002$......

This table is easily formed by dividing 1.00000000 by 2, the result by 3, the result so obtained by 4, and so on.

Sum to seven decimal places $= .7182818$.

Hence if we consider only the first seven decimal places, 2.7182818 is the limiting value of $\left(1+\frac{1}{n}\right)^n$, when n approaches ∞.

50. In the same manner we might obtain this limiting value to any greater number of decimal places. It is, however, an incommensurable number like π, and therefore, no matter how far we go, we shall never be able to obtain it exactly in numbers. It is usually denoted by the letter e, and it is taken as the base of a system of logarithms commonly called *Hyperbolic*. Base of Hyperbolic Logarithms; what.

For a certain reason, which we shall presently explain, logarithms calculated to base e are the most convenient to use in analytical calculations. We shall therefore always suppose that e is the base of whatever logarithms we have occasion to employ unless the contrary be specified.

51. *Hence* e *is the limiting value of* $(1+x)^{\frac{1}{x}}$ *when* x *approaches zero,* x *being any continuous variable.* Cor. 1.

For this limiting value by Lemma I. is the same thing as the limiting value of $(1+z)^{\frac{1}{z}}$ when z approaches 0, where $z=\frac{1}{n}$ and n is always an integer; and this latter limiting value by Lemma II. is the same thing as that of $\left(1+\frac{1}{n}\right)^n$ when n approaches ∞, which $=e$. Q. E. D.

52. *Hence,* $\log_a e$ *or* $\frac{1}{\log a}$[*] *is the limiting value of* $\frac{\log_a(1+x)}{x}$ *when* x *approaches zero.* Cor. 2.

For $\frac{\log_a(1+x)}{x}=\log_a(1+x)^{\frac{1}{x}}$, and by what has just been proved, and by Lemma VIII. the limiting value of this latter quantity when x approaches zero is $\log_a e$.

---

[*] If $\log_a e=c$, then $a^c=e$, and $\therefore c\log_e a=1$, or $c=\frac{1}{\log a}$.

Cor. 3. 53. *Hence also,* $\frac{1}{\log_a e}$ *or log* a *is the limiting value of* $\frac{a^x - 1}{x}$ *when* x *approaches zero.*

For, put $a^x - 1 = z$, and $\therefore\ x = \log_a (1 + z)$;

$$\text{then } \frac{a^x - 1}{x} = \frac{z}{\log_a (1 + z)} = \frac{1}{\log_a (1 + z)^{\frac{1}{z}}}.$$

Now $x = 0$ when $z = 0$; hence by Lemma II, the limiting value of $\frac{a^x - 1}{x}$ when x approaches 0, is the same thing as that of $\frac{1}{\log_a (1 + z)^{\frac{1}{z}}}$ when z approaches 0 which, by what has been just proved, $= \frac{1}{\log_a e}$ or $\log a$.

CHAPTER V.

RULES FOR FINDING THE LIMITING VALUE OF $\dfrac{f(x')-f(x)}{x'-x}$.

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54. We now enter upon the Differential Calculus, properly so called, which, as we have stated, is that branch of mathematics whose object is, in the first place, to determine a set of Rules whereby the limiting value of $\dfrac{f(x')-f(x)}{x'-x}$ when x' approaches x may be found in all cases with facility; and in the second place, to shew and explain some of the principal uses which may be made of this limiting value in pure mathematics.

We proceed therefore, in the first place, to determine a set of Rules whereby this limiting value may be found in all cases with facility.

Notation by which we represent the limiting value of $\dfrac{f(x')-f(x)}{x'-x}$

55. *We shall represent the limiting value of* $\dfrac{\mathrm{f}(\mathrm{x}')-\mathrm{f}(\mathrm{x})}{\mathrm{x}'-\mathrm{x}}$ *when* x' *approaches* x *by the notation* $\mathrm{f}'(\mathrm{x})$*; i. e. by simply dashing the functional letter.*

Examples.

Thus if $f(x)=x^2$, $\dfrac{f(x')-f(x)}{x'-x}=\dfrac{x'^2-x^2}{x'-x}=x'+x$,

except when x' actually $=x$; now the actual value of $x'+x$ when $x'=x$, is $2x$; hence, by Lemma VI., the limiting value of $\dfrac{x'^2-x^2}{x'-x}$ when x' approaches x is $2x$; i. e.

$$\text{if } f(x)=x^2, \text{ then } f'(x)=2x.$$

Again if $f(x)=ax+bx^3$, $\dfrac{f(x')-f(x)}{x'-x}$

$$=\frac{a(x'-x)+b(x'^3-x^3)}{x'-x}=a+b(x'^2+x'x+x^2);$$

hence, as in the previous case, we find that if

$$f(x) = ax + bx^3, \quad \text{then } f'(x) = a + 3bx^2.$$

And in general we may very easily shew in the same manner that if

$$f(x) = a_0 + a_1x + a_2x^2 \ldots a_nx^n,$$

$$\text{then } f'(x) = a_1 + 2a_2x + 3a_3x^2 \ldots + na_nx^{n-1}.$$

Thus when $f(x)$ is a rational and integral function of x, $f'(x)$ may be *derived* from it by multiplying each power of x by its index, and lowering that index by unity.

Origin of the notation $f'(x)$, and of the name Derivative or Derived Function.

56. This notation f' is due to Lagrange: the function $f'(x)$ was called by him the *Derived Function* or *Derivative* of $f(x)$, because it may be derived from $f(x)$ by the peculiar process just stated, at least when $f(x)$ is a rational and integral function of x.

The following are the Rules whereby we may find this derived function or derivative in all cases with facility.

Rule I. $f(x) = c$.

57. *If* f(x) = *a constant* c*, f′(x) = 0.

For then $\dfrac{f(x') - f(x)}{x' - x} = \dfrac{c - c}{x' - x} = 0$, except $x' = x$;

$\therefore$ lim. val. of $\dfrac{f(x') - f(x)}{x' - x}$ when x' app. x is zero, by Lem. VI.,

$$\text{i.e. } f'(x) = 0.$$

Cor.

If $f(x) = c + \phi(x)$, $f'(x) = \phi'(x)$.

For then $\dfrac{f(x') - f(x)}{x' - x} = \dfrac{\phi(x') - \phi(x)}{x' - x}$,

and $\therefore f'(x) = \phi'(x)$. by Lem. VI. Cor. 1.

A constant therefore added to a function does not appear in the derived function.

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[*] We have seen in (5) that a function of x may be a constant quantity.

58. *If* f(x) = cϕ(x), ϕ *denoting any function; then* f′(x) = cϕ'(x). Rule II. $f(x) = c\phi(x)$.

For then $\dfrac{f(x') - f(x)}{x' - x} = c\dfrac{\phi(x') - \phi(x)}{x' - x}$,

the limiting value of the second member of this equation when x' approaches x is $c\phi'(x)$: hence, by Lem. VI. Cor. 1,

$$f'(x) = c\phi'(x).$$

59. *If* f(x) = ϕ(x) ± ψ(x) ± χ(x) ± &c., ϕ, ψ, χ, &c., *denoting any functions; then* Rule III. $f(x) = \phi(x) \pm \psi(x)$...

$$f'(x) = \phi'(x) \pm \psi'(x) \pm \chi'(x) \pm \&c.$$

For then
$$\frac{f(x') - f(x)}{x' - x} = \frac{\phi(x') - \phi(x)}{x' - x} \pm \frac{\psi(x') - \psi(x)}{x' - x}$$
$$\pm \frac{\chi(x') - \chi(x)}{x' - x} \pm \ldots \&c.$$

The limiting value of the second member of this equation when x' approaches x is

$\phi'(x) \pm \psi'(x) \pm \chi'(x) \pm \ldots$ &c., by Lemma VIII.;

hence, $f'(x) = \phi'(x) \pm \psi'(x) \pm \chi'(x) \pm \ldots$ &c.

60. *If* f(x) = ϕ(x) . ψ(x), *then* Rule IV. $f(x) = \phi(x)\psi(x)$.

$$f'(x) = \phi'(x) . \psi(x) + \phi(x) . \psi'(x).$$

For then $\dfrac{f(x') - f(x)}{x' - x}$

$$= \frac{\phi(x')\psi(x') - \phi(x)\psi(x') + \phi(x)\psi(x') - \phi(x)\psi(x)}{x' - x}$$

$$= \frac{\phi(x') - \phi(x)}{x' - x}\psi(x') + \phi(x) . \frac{\psi(x') - \psi(x)}{x' - x};$$

now the limiting value of the second member of this equation when x' approaches x is $\phi'(x)\psi(x) + \phi(x)\psi'(x)$, by Lemma VIII.;

hence, $f'(x) = \phi'(x)\psi(x) + \phi(x)\psi'(x)$.

Cor. 1. $f(x) = \frac{\phi(x)}{\psi(x)}$.

61. *If* $\mathrm{f}(\mathrm{x}) = \frac{\phi(\mathrm{x})}{\psi(\mathrm{x})}$, *then*

$$\mathrm{f}'(\mathrm{x}) = \frac{\phi'(\mathrm{x})\,\psi(\mathrm{x}) - \phi(\mathrm{x})\,\psi'(\mathrm{x})}{\{\psi(\mathrm{x})\}^2}.$$

For then $f(x)\,\psi(x) = \phi(x)$; and, as in the rule just proved, we have

$$f'(x)\,\psi(x) + f(x)\,\psi'(x) = \phi'(x);$$

$$\text{whence}\quad f'(x) = \frac{\phi'(x) - f(x)\,\psi'(x)}{\psi(x)}$$

$$= \frac{\phi'(x)\,\psi(x) - \phi(x)\,\psi'(x)}{\{\psi(x)\}^2},$$

putting for $f(x)$ its value.

Rule V. $f(x) = x^n$.

62. *If* $\mathrm{f}(\mathrm{x}) = \mathrm{x}^{\mathrm{n}}$, *then* $\mathrm{f}'(\mathrm{x}) = \mathrm{n}\,\mathrm{x}^{\mathrm{n}-1}$; *whether* n *be integral or fractional, positive or negative.*

$$\text{For then}\quad \frac{f(x') - f(x)}{x' - x} = \frac{x'^n - x^n}{x' - x}$$

$$= x^{n-1}\,.\,\frac{z^n - 1}{z - 1}\ \text{if we put}\ x' = xz.$$

Now $x' = x$ when $z = 1$, therefore by Lemma II. the limiting value of the second member of this equation when z approaches 1 is the same thing as that of the first member when x' approaches x; but by Lemma XI. n is the limiting value of $\frac{z^n - 1}{z - 1}$ when z approaches 1. Hence $f'(x) = nx^{n-1}$.

Examples of the use of these five rules.

63. By means of these five Rules we may find the derivative of any rational function of x. For example:

Let $\quad f(x) = a_0 + a_1 x + a_2 x^2 \ldots\ldots a_n x^n,$

then by Rule III. $f'(x)$ is found by taking the derivative of each term separately; also by Rule I. the derivative of a_0 is 0, and by Rule II. the derivative of any other term $a_m x^m$ is $a_m \times$ the derivative of x^m, or $a_m m x^{m-1}$ by Rule V.

Hence $f'(x) = a_1 + 2a_2x + 3a_3x^2 \ldots\ldots na_nx^{n-1}$.

Again, let $f(x) = \dfrac{x^2 - a^2}{x^2 + a^2} = \dfrac{\phi(x)}{\psi(x)}$ suppose;

then $\phi'(x) = 2x$, $\psi'(x) = 2x$, and therefore by Cor. Rule IV.

$$f'(x) = \frac{2x(x^2 + a^2) - (x^2 - a^2)\,2x}{(x^2 + a^2)^2} = \frac{4a^2x}{(x^2 + a^2)^2}.$$

Again, let $f(x) = Ax^{\frac{5}{2}} + Bx^{-3}$,

then $f'(x) = \frac{5}{2}Ax^{\frac{3}{2}} - 3Bx^{-4}$ by Rules II. III. V.

Again, let

$f(x) = (x^2 + x^{-2})(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) = \phi(x)\,.\,\psi(x)$ suppose;

then $\phi'(x) = 2x - 2x^{-3}$, $\psi'(x) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$;

hence by Rule IV.

$$f'(x) = 2(x - x^{-3})(x^{\frac{1}{2}} + x^{-\frac{1}{2}}) + \tfrac{1}{2}(x^2 + x^{-2})(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}).$$

(For more examples see Appendix A.)

64. *If* $\mathrm{f}(\mathrm{x}) = log_{\mathrm{a}}\mathrm{x}$, *then* $\mathrm{f}'(\mathrm{x}) = \dfrac{1}{log\,\mathrm{a}}\cdot\dfrac{1}{\mathrm{x}}$.

Rule VI.
$f(x) = \log_a x$.

For then

$$\frac{f(x') - f(x)}{x' - x} = \frac{\log_a x' - \log_a x}{x' - x}$$

$$= \frac{1}{x}\,\frac{\log_a(xz) - \log_a x}{z - 1} \text{ if we put } x' = xz$$

$$= \frac{1}{x}\,\frac{\log_a z}{z - 1}.$$

Now $x' = x$ when $z = 1$, therefore, by Lemma II., and Lemma XIII. Cor. 2, $\dfrac{1}{\log a}\cdot\dfrac{1}{x}$ is the limiting value of $\dfrac{f(x') - f(x)}{x' - x}$ when x' approaches x, i.e. $f'(x) = \dfrac{1}{\log a}\cdot\dfrac{1}{x}$.

Cor. $f(x)=\log x$.

If $a=e$, then $\log a = 1$, which gives $f'(x) = \dfrac{1}{x}$.

Reason why e is usually taken as the base of logarithms.

It appears therefore that we get a very simple expression for the derivative of the logarithm of x when e is taken as the base. It is for this reason that this base is chosen in all analytical investigations, except when numerical calculations come in, in which case 10 is a much more convenient base.

Rule VII. $f(x)=a^x$.

65. *If* $\mathrm{f}(\mathrm{x}) = \mathrm{a}^{\mathrm{x}}$, *then* $\mathrm{f}'(\mathrm{x}) = \log \mathrm{a} \, . \, \mathrm{a}^{\mathrm{x}}$.

For then

$$\frac{f(x') - f(x)}{x' - x} = \frac{a^{x'} - a^x}{x' - x}$$

$$= a^x \, . \, \frac{a^z - 1}{z} \text{ if we put } x' - x = z.$$

Now $x' = x$ when $z = 1$, therefore, by Lemma II., and by Lemma XIII. Cor. 3, $a^x \log a$ limiting value of $\dfrac{f(x')-f(x)}{x' - x}$ when x' approaches x. Hence $f'(x) = \log a \, . \, a^x$.

Cor. $f(x)=\epsilon^x$.

If $f(x) = e^x$, then $f'(x) = \epsilon^x$, since $\log_e e = 1$.

Rule VIII. $f(x)=\sin x$ or $\cos x$.

66. *If* $\mathrm{f}(\mathrm{x}) = \sin \mathrm{x}$, *then* $\mathrm{f}'(\mathrm{x}) = \cos \mathrm{x}$; *and if* $\mathrm{f}(\mathrm{x}) = \cos \mathrm{x}$, *then* $\mathrm{f}'(\mathrm{x}) = -\sin \mathrm{x}$.

For if $f(x) = \sin x$

$$\frac{f(x') - f(x)}{x' - x} = \frac{\sin x' - \sin x}{x' - x}$$

$$= \frac{2 \cos \dfrac{x' + x}{2} \, . \, \sin \dfrac{x' - x}{2}}{x' - x}$$

$$= \cos(x + z) \frac{\sin z}{z} \text{ if we put } \frac{x' - x}{2} = z.$$

Now the limiting value of the second member of this equation when z approaches 0 is $\cos x$, by Lemma VIII. and

Lemma X.; therefore, since $x' = x$ when $z = 0$, we have by Lemma II. $f'(x) = \cos x$.

And if $f(x) = \cos x$ we find similarly

$$\frac{f(x') - f(x)}{x' - x} = -\frac{2 \sin \dfrac{x' + x}{2} \sin \dfrac{x' - x}{2}}{x' - x}$$

$$= -\sin (x + z) \sin z\,;$$

and therefore $f'(x) = -\sin x$.

Hence if $f(x) = \operatorname{cosec} x = \dfrac{1}{\sin x}$, $f'(x) = -\dfrac{\cos x}{\sin^2 x}$, by Rule IV. Cor. 1, putting $\phi(x) = 1$ and $\psi(x) = \sin x$.

Cor. $f(x) = \sec x$ or $\operatorname{cosec} x$.

And if $f(x) = \sec x = \dfrac{1}{\cos x}$, $f'(x) = \dfrac{\sin x}{\cos^2 x}$ similarly.

67. *If* $f(x) = \tan x$, *then* $f'(x) = \dfrac{1}{\cos^2 x}$, *and if* $f(x) = \cot x$, *then* $f'(x) = -\dfrac{1}{\sin^2 x}$.

Rule IX. $f(x) = \tan x$ or $\cot x$.

For if $f(x) = \tan x$,

$$\frac{f(x') - f(x)}{x' - x} = \frac{\tan x' - \tan z}{x' - x}$$

$$= \frac{1}{\cos x' \cos x} \cdot \frac{\sin (x' - x)}{x' - x}$$

$$= \frac{1}{\cos (x + z) \cos x} \cdot \frac{\sin z}{z} \text{ if we put } x' - x = z.$$

Hence, as in the former case, we have $f'(x) = \dfrac{1}{\cos^2 x}$.

And if $f(x) = \cot x$,

$$\frac{f(x') - f(x)}{x' - x} = -\frac{1}{\sin x' \sin x} \cdot \frac{\sin (x' - x)}{x' - x}\,;$$

and therefore $f'(x) = -\dfrac{1}{\sin^2 x}$.

Cor. This rule deducible from the preceding.

This Rule may easily be deduced from the preceding, as follows:

If $f(x) = \tan x = \frac{\sin x}{\cos x}$ then by Rule IV. Cor. and by Rule VIII,

$$f'(x) = \frac{\cos x \cos x - \sin x(-\cos x)}{\cos^2 x} = \frac{1}{\cos^2 x}$$

and if $f(x) = \cot x = \frac{\cos x}{\sin x}$,

$$\text{then } f'(x) = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

Rule X. $f(x) = \sin^{-1} x$ or $\cos^{-1} x$.

68. *If* $f(x) = \sin^{-1} x$, *then* $f'(x) = \frac{1}{\sqrt{1 - x^2}}$, *and if* $f(x) = \cos^{-1} x$, *then* $f'(x) = -\frac{1}{\sqrt{1 - x^2}}$.

For if $f(x) = \sin^{-1} x$, and therefore $x = \sin f(x)$, we have

$$\frac{f(x') - f(x)}{x' - x} = \frac{f(x') - f(x)}{\sin f(x') - \sin f(x)}$$

$$= \frac{f(x') - f(x)}{2 \cos \frac{f(x') + f(x)}{2} \cdot \sin \frac{f(x') - f(x)}{2}}$$

$$= \frac{1}{\cos \{f(x) + z\}} \cdot \frac{z}{\sin z},$$

if we put $\frac{f(x') - f(x)}{2} = z$.

Now $x' = x$ when $z = 0$; hence, by Lemmas II. and X. we have

$$f'(x) = \frac{1}{\cos f(x)} = \frac{1}{\sqrt{1 - x^2}}, \text{ since } x = \sin f(x).$$

And similarly, we may shew that if $f(x) = \cos^{-1} x$,

$$f'(x) = -\frac{1}{\sqrt{1 - x^2}}.$$

69. *If* $f(x) = \tan^{-1} x$, *then* $f'(x) = \frac{1}{1+x^2}$, *and if* $f(x) = \cot^{-1} x$, *then* $f'(x) = -\frac{1}{1+x^2}$.

Rule XI. $f(x)=\tan^{-1}x$ or $\cot^{-1}x$.

For if $f(x) = \tan^{-1} x$,

$$\frac{f(x')-f(x)}{x'-x} = \frac{f(x')-f(x)}{\tan f(x') - \tan f(x)}$$

$$= \cos f(x') \cos f(x) \cdot \frac{f(x')-f(x)}{\sin\{f(x')-f(x)\}}$$

$$= \cos\{f(x)+z\}\cos f(x) \cdot \frac{z}{\sin z},$$

if we put $f(x') - f(x) = z$.

Hence, as before,

$$f'(x) = \cos^2 f(x) = \frac{1}{1+x^2}, \text{ since } \tan f(x) = x.$$

And similarly, we may shew that if $f(x) = \cot^{-1}(x)$, $f'(x) = -\frac{1}{1+x^2}$.

We have proved Rules VII, X, XI, directly, but they may be deduced very simply from Rules VI, VIII, IX, by means of the following Rule, as we shall shew.

70. *If we are given* $x = \phi(y)$, *in which case* y *will be some function of* x, $f(x)$ *suppose; then* $f'(x) = \frac{1}{\phi'(y)}$.

Rule XII. If $x=\phi(y)$ and thence $y=f(x)$ to find $f'(x)$.

For then $\frac{f(x')-f(x)}{x'-x} = \frac{y'-y}{\phi(y')-\phi(y)}$,

$$= \frac{1}{\dfrac{\phi(y')-\phi(y)}{y'-y}}.$$

Now, by Lemma VIII, the limiting value of the second member of this equation when y' approaches y is $\frac{1}{\phi'(y)}$, and $x' = x$ when $y' = y$; hence, by Lemma II,

$$f'(x) = \frac{1}{\phi'(y)}.$$

Rules VII. X. XI. deduced from Rules VI. VIII. IX. by means of Rule XII.

71. This Rule is of great use in finding the derivatives of inverse functions, and by means of it may deduce Rules VII, X, XI, from Rules VI, VIII, IX, respectively, as follows:

If $x = \log_a y = \phi(y)$ suppose, then $y = a^x = f(x)$ suppose; and therefore, since

$$f'(x) = \frac{1}{\phi'(y)}, \text{ and } \phi'(y) = \frac{1}{\log a}\frac{1}{y}, \text{ by Rule VI,}$$

we have $f'(x) = \log a \,.\, y = \log a \; a^x$; which is Rule VII.

If $x = \sin y = \phi(y)$ suppose, then $y = \sin^{-1} x = f(x)$ suppose; and therefore, since

$$f'(x) = \frac{1}{\phi'(y)} = \frac{1}{\cos y}, \text{ by Rule VIII,}$$

we have $f'(x) = \dfrac{1}{\sqrt{1 - x^2}}$, since $\cos y = \sqrt{1 - x^2}$; which is Rule X.

If $x = \tan y = \phi(y)$ suppose, then $y = \tan^{-1} x = f(x)$ suppose; and we have therefore

$f'(x) = \dfrac{1}{\phi'(y)} = \cos^2 y$, by Rule IX. which $= \dfrac{1}{1 + x^2}$, since $\tan y = x$; which is Rule XI.

Rule XII. applied to the case of $\sec^{-1} x$ $\text{vers}^{-1} x$.

72. We may also by this Rule obtain the following derivatives which it is sometimes useful to remember.

If $x = \sec y = \phi(y)$, then $y = \sec^{-1} x = f(x)$, and we have therefore

$$f'(x) = \frac{1}{\phi'(x)} = \frac{\cos^2 y}{\sin y}, \text{ by Cor. Rule VIII.} = \frac{1}{x\sqrt{x^2 - 1}},$$

since $\cos y = \dfrac{1}{x}$.

Hence, if $f(\mathrm{x}) = \sec^{-1} \mathrm{x}$, $\quad f'(\mathrm{x}) = \dfrac{1}{\mathrm{x}\sqrt{\mathrm{x}^2 - 1}}$.

Again, if $x = \text{vers}\, y = \phi(y)$, then $y = \text{vers}^{-1} x = f(x)$, and we have therefore

$$f'(x) = \frac{1}{\phi'(y)} = \frac{1}{\sin y}, \text{ by Rule VIII, since } \operatorname{vers} y = 1 - \cos y$$

$$= \frac{1}{\sqrt{2x - x^2}}, \text{ since } \cos y = 1 - x.$$

Hence, if $\mathrm{f}(\mathrm{x}) = vers^{-1}\,\mathrm{x}$, $\mathrm{f}'(\mathrm{x}) = \dfrac{1}{\sqrt{2\mathrm{x} - \mathrm{x}^2}}$.

73. *If we are given two relations between* x *and* y *by the intervention of a third variable* v *in the form*

$$\mathrm{y} = \phi(\mathrm{v}), \quad \textit{and} \quad \mathrm{v} = \psi(\mathrm{x}),$$

in which case y *will be some function of* x, $\mathrm{f}(\mathrm{x})$ *suppose; then* $\mathrm{f}'(\mathrm{x}) = \phi'(\mathrm{v})\,\psi'(\mathrm{x})$.

Rule XIII. If $y = \phi(v)$ $v = \psi(x)$, and thence $y = f(x)$ to find $f'(x)$.

For then we have

$$\frac{f(x') - f(x)}{x' - x} = \frac{\phi(v') - \phi(v)}{v' - v} \cdot \frac{\psi(x') - \psi(x)}{x' - x}.$$

Now, since $v' = v$ when $x' = x$, the limiting value of $\dfrac{\phi(v') - \phi(v)}{v' - v}$ when x' approaches x, is the same (by Lem. II.) as that of $\dfrac{\phi(v') - \phi(v)}{v' - v}$ when v' approaches v, which $= \phi'(v)$; hence, by Lem. VIII, the limiting value of the second member of this equation, when x' approaches x, is $\phi'(v)\,\psi'(x)$; and therefore $f'(x) = \phi'(v)\,\psi'(x)$.

74. In the same manner we may shew, that if $y = \phi(v)$, $v = \psi(u)$, $u = \chi(x)$, and therefore y some function of x, $f(x)$ suppose; then $f'(x) = \phi'(v)\,\psi'(u)\,\chi'(x)$. And a similar result if any number of variables intervene between y and x.

Cor. Any number of variables intervening between y and x.

For as before we have

$$\frac{f(x') - f(x)}{x' - x} = \frac{\phi(v') - \phi(v)}{v' - v} \cdot \frac{\psi(u') - \psi(u)}{u' - u} \cdot \frac{\chi(x') - \chi(x)}{x' - x};$$

$$\text{and } \therefore f'(x) = \phi'(v) \cdot \psi'(u) \cdot \chi'(x).$$

Examples of the use of this Rule. 75. This Rule is of great use in finding the derivatives of complex functions, as the following examples will shew.

$$\text{Let } f(x) = \phi(c + x), \quad \text{put } c + x = v = \psi(x),$$
$$\text{and } \therefore \psi'(x) = 1;$$
$$\text{then } f'(x) = \phi'(v) = \phi'(c + x).$$

Hence the derivative of $\phi(c + x)$ is $\phi'(c + x)$. Thus the derivatives of $(c + x)^n$, $\log(c + x)$, e^{c+x}, $\sin(c + x)$, &c., are respectively, $n(c + x)^{n-1}$, $\dfrac{1}{c + x}$, e^{c+x}, $\cos(c + x)$, &c., by Rules V, VI, VII, VIII.

$$\text{Let } f(x) = \phi(cx), \text{ put } cx = v = \psi(x), \text{ and } \therefore \psi'(x) = c,$$
$$\text{then } f'(x) = \phi'(v)\,c = c\phi'(cx).$$

Hence the derivative of $\phi(cx)$ is $c\phi'(cx)$. Thus the derivatives of e^{cx}, $\tan^{-1}\dfrac{x}{c}$, $\sin^{-1}\dfrac{x}{c}$, are respectively

$$ce^{cx}, \quad \frac{c}{c^2 + x^2}, \quad \frac{1}{\sqrt{c^2 - x^2}}, \text{ by Rules VII, X, XI.}$$

$$\text{Let } f(x) = \phi(x^n), \quad \text{put } x^n = v = \psi(x),$$
$$\text{and } \therefore \psi'(x) = ux^{n-1} \quad \text{(Rule V.)},$$
$$\text{then } f'(x) = \phi'(v)\,nx^{n-1} = n\,\phi'(x^n)\,x^{n-1}.$$

Hence the derivative of $\phi(x^n)$, is $n\,\phi'(x^n)\,x^{n-1}$.

$$\text{Let } f(x) = e^{e^x}, \quad \text{put } v = e^x = \psi(x),$$
$$\text{and } \therefore f(x) = e^v = \phi(v);$$
$$\text{then } f'(x) = \phi'(v)\,\psi'(x) = e^v e^x \text{ (Rule VII.)} = e^{e^x} . e^x = e^{x+e^x}.$$

$$\text{Let } f(x) = e^{x+e^x}, \quad \text{put } v = x + e^x = \psi(x),$$
$$\text{and } \therefore f(x) = e^v = \phi(v);$$
$$\text{then } f'(x) = \phi'(v)\,\psi'(x) = e^v(1 + e^x) \text{ (Rules VII, III.)}$$
$$= e^{e^x}(1 + e^x).$$

Let $f(x) = \log \psi(x)$, put $v = \psi(x)$,

$$\text{and } \therefore f(x) = \log v = \phi(v);$$

$$\text{then } f'(x) = \phi'(v)\,\psi'(x) = \frac{1}{v}\psi'(x) = \frac{\psi'(x)}{\psi(x)}.$$

(For more Examples see Appendix B.)

Rule XIV. The derivative of a complicated product found by logarithims.

76. *The derivative of a complicated product may often be conveniently found by taking its logarithm first.*

$$\text{Let } f(x) = \{\phi(x)\}^m.\{\psi(x)\}^n.\{\chi(x)\}^p \ldots$$

$$\text{then } \log f(x) = m \log \phi(x) + n \log \psi(x) + p \log \chi(x) \ldots$$

hence, taking the derivative by Rules XIII, III, and II,

$$\frac{f'(x)}{f(x)} = m\frac{\phi'(x)}{\phi(x)} + n\frac{\psi'(x)}{\psi(x)} + p\frac{\chi'(x)}{\chi(x)} \ldots$$

which gives $f'(x)$ very readily, as the following example will shew.

Example.

$$\text{Let } f(x) = e^x\sqrt{\frac{1+x}{1-x}};$$

$$\text{then } \log f(x) = x + \tfrac{1}{2}\log(1+x) - \tfrac{1}{2}\log(1-x);$$

$$\therefore \frac{f'(x)}{f(x)} = 1 + \tfrac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) = 1 + \frac{1}{1-x^2} = \frac{2-x^2}{1-x^2};$$

$$\therefore f'(x) = e^x\sqrt{\frac{1+x}{1-x}}.\frac{2-x^2}{1-x^2}.$$

$$\text{Let } f(x) = \frac{\sin x.\sqrt{1-x^2}}{x^2};$$

$$\text{then } \log f(x) = \log \sin x + \tfrac{1}{2}\log(1-x^2) - 2\log x;$$

$$\therefore \frac{f'(x)}{f(x)} = \cot x + \frac{-x}{1-x^2} - \frac{2}{x} = \cot x - \frac{2-x^2}{x(1-x^2)};$$

$$\therefore f'(x) = \frac{\sin x\sqrt{1-x^2}}{x^2}\left\{\cot x - \frac{2-x^2}{x(1-x^2)}\right\}.$$

Let $f(x) = x^x$; then $\log f(x) = x \log x$, and therefore we have

$$\frac{f'(x)}{f(x)} = \log x + x \cdot \frac{1}{x};$$

$$\therefore\ f'(x) = x^x(\log x + 1).$$

††77. *To these Rules we must add two more, which will be found of great use in many cases, and important hereafter, when we come to consider more than one independant variable. But we must first explain a certain notation which we shall make use of.

Partial derivatives; what. The notation $\phi'(\underset{\cdot}{x}y)$ and $\phi'(x\underset{\cdot}{y})$.

If $f(xy)$ be any function of two variables x and y, we shall assume $f'(\underset{\cdot}{x}y)$ to denote the limiting value of $\dfrac{f(x'y) - f(xy)}{x' - x}$ when x' approaches x. It is evident that this limiting value is the derived function of $f(xy)$, taken on the supposition that y is constant and x alone variable. It is called the *Partial Derivative* of $f(xy)$ with respect to x.

In the same manner we shall take $f'(x\underset{\cdot}{y})$ to denote the limiting value of $\dfrac{f(xy') - f(xy)}{y' - y}$, when y' approaches y; i. e. *the Partial Derivative* of $f(xy)$ with respect to y, or the derivative taken on the supposition that y alone is variable.

Examples.

Thus if $f(xy) = x^2 + xy + y^2$, then

$$f'(\underset{\cdot}{x}y) = 2x + y, \text{ and } f'(x\underset{\cdot}{y}) = y + 2x.$$

Again, if $f(xy) = x \sin y + ye^x$,

$$f'(\underset{\cdot}{x}y) = \sin y + ye^x, \text{ and } f'(x\underset{\cdot}{y}) = x \cos y + e^x.$$

Thus by dashing the functional letter we denote a derivative of a function of two variables, and by a dot under one of the variables we signify that *that* variable alone is supposed to vary.

This notation does not necessarily require that

In using this notation we do not necessarily assume that y is independant of x; for whether y be independant of x or not, we may always put x' in place of x if we please, and so find $f(x'y)$, and thence $\dfrac{f(x'y)-f(xy)}{x'-x}$; and then, by the previous Rules, we may determine the limiting value of this quantity when x' approaches x. And the same may be said of $f'(xy)$. The following is the Rule for which it is necessary to make use of this notation or something equivalent to it. x and y should be independant of each other.

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* The Articles marked thus †† may be omitted till they are referred to, at least on a first perusal.

††78. *If we are given a relation between* y *and two other variables* u *and* v *which are functions of* x*; i. e. if* Rule XV.

$$y=\phi(uv),\quad u=\psi(x),\quad v=\chi(x),$$

in which case y *will be some function of* x, f(x) *suppose; then*

$$f'(x)=\phi'(uv)\,\psi'(x)+\phi'(uv)\,\chi'(x).$$

For $\dfrac{f(x')-f(x)}{x'-x}=\dfrac{\phi(u'v')-\phi(uv')}{x'-x}+\dfrac{\phi(uv')-\phi(uv)}{x'-x}$,

and $\dfrac{\phi(u'v')-\phi(uv')}{x'-x}=\dfrac{\phi(u'v')-\phi(uv')}{u'-u}\cdot\dfrac{\psi(x')-\psi(x)}{x'-x}$

$$\frac{\phi(uv')-\phi(uv)}{x'-x}=\frac{\phi(uv')-\phi(uv)}{v'-v}\cdot\frac{\chi(x')-\chi(x)}{x'-x}.$$

Now when x' approaches x, the limiting values of

$$\frac{\psi(x')-\psi(x)}{x'-x},\quad \frac{\phi(uv')-\phi(uv)}{v'-v},\ \text{and}\ \frac{\chi(x')-\chi(x)}{x'-x},$$

are respectively $\psi'(x)$, $\phi'(uv)$, and $\chi'(x)$. Also since v' becomes v when $x'=x$, the limiting value of $\dfrac{\phi(u'v')-\phi(uv')}{u'-u}$*

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* If we suppose for a moment that v' is not $\chi(x')$ but some constant, then $\phi'(uv')$ is the limiting value of $\dfrac{\phi(u'v')-\phi(uv')}{u'-u}$ when x' approaches x: therefore $\dfrac{\phi(u'v')-\phi(uv')}{u'-u}\sim\phi'(uv')$ may be diminished *ad libitum* by sufficiently diminishing $x'\sim x$; and this is evidently true whatever be the value of v'; therefore it must be true if we suppose v' to vary in any manner while we diminish $x'\sim x$; it is therefore true if $v'=\chi(x')$, and therefore by Lemma VI. Cor. 2, $\phi'(uv)$, which is the limiting value of $\phi'(uv')$, is also that of $\dfrac{\phi(u'v')-\phi(uv')}{u'-u}$, supposing $v'=\chi(x')$.

is evidently the same thing as that of $\dfrac{\phi(u'v) - \phi(uv)}{u' - u}$, which is $\phi'(\underset{\cdot}{u}v)$. Hence we have

$$f'(x) = \phi'(\underset{\cdot}{u}v)\,\psi'(x) + \phi'(u\underset{\cdot}{v})\,\chi'(x).$$

Examples. Let $\phi(uv) = u^v$,

$$\text{then}\quad \phi'(\underset{\cdot}{u}v) = vu^{v-1} \quad\text{and}\quad \phi'(u\underset{\cdot}{v}) = \log u\, u^v;$$

$$\therefore\ f'(x) = vu^{v-1}\psi'(x) + \log u\, u^v \chi'(x).$$

Suppose that $u = x$ and $v = x$, and $\therefore\ \psi'(x) = 1\ \chi'(x) = 1$; then $f'(x) = x^x + \log x\, x^x = x^x(1 + \log x)$.

Again, let $\phi(uv) = u^2 + v^2 - uv$;

$$\text{then}\quad \phi'(\underset{\cdot}{u}v) = 2u - v\quad \phi'(u\underset{\cdot}{v}) = 2v - u;$$

$$\therefore\ f'(x) = (2u - v)\,\psi'(x) + (2v - u)\,\chi'(x).$$

Cor. If $y = \phi(uvw)$ w being another function of x, $\zeta(x)$, suppose then by putting $f(x') - f(x)$ in the form

$$\{\phi(u'v'w') - \phi(uv'w')\} + \{\phi(uv'w') - \phi(uvw')\}$$
$$+ \{\phi(uvw') - \phi(uvw)\}.$$

We may shew exactly as before, that

$$f'(x) = \phi'(\underset{\cdot}{u}vw)\,\psi'(x) + \phi'(u\underset{\cdot}{v}w)\,\chi'(x) + \phi'(uv\underset{\cdot}{w})\,\zeta'(x).$$

This Rule we may evidently extend to the case where y is a function of any number of functions of x.

Rule XVI. ††79. *If an equation be given between* x *and* y, *which of course makes* y *a function of* x, *we may find the derivative of* y *by means of Rule* XV.

For, let the given equation be

$$\phi(yx) = 0;$$

in virtue of this equation $y =$ some function of x, $\psi(x)$ suppose; and $\phi(yx)$, by substituting for y this value, becomes also a function of x, $f(x)$ suppose: then, by Rule XV, putting y for u, and v for x, and therefore $\chi'(x) = 1$, we have

$$f(x) = \phi'(\underset{\cdot}{y}x)\,\psi'(x) + \phi'(y\underset{\cdot}{x}).$$

Now, $f(x) = 0$, and therefore by Rule I, $f'(x) = 0$; therefore we have

$$\phi'(\underset{\cdot}{y}\,x)\,\psi'(x) + \phi'(y\,\underset{\cdot}{x}) = 0\,;$$

$$\text{or,}\quad \psi'(x) = -\frac{\phi'(y\,\underset{\cdot}{x})}{\phi'(\underset{\cdot}{y}\,x)}\,.$$

Example.

Let the given equation be

$$x^2 + y^2 - axy + b = 0,$$

$$\text{here,}\quad \phi'(\underset{\cdot}{y}\,x) = 2y - ax, \qquad \phi'(y\,\underset{\cdot}{x}) = 2x - ay;$$

$$\therefore\ \text{if}\ \psi(x) = y, \quad (2y - ax)\,\psi'(x) + 2x - ay = 0\,;$$

$$\text{or,}\ \psi'(x) = -\frac{2x - ay}{2y - ax}\,.$$

CHAPTER VI.

THE DIFFERENTIAL NOTATION.

80. We have hitherto adopted Lagrange's notation, $f'(x)$, to represent the limiting value of $\frac{f(x')-f(x)}{x'-x}$ when x' approaches x; this notation is often very convenient, and is perhaps more easily understood than any other, at least when the student enters upon this subject for the first time; and this is the reason why we have used it in obtaining the preceding Rules. But there is a far more elegant and powerful notation, due to Leibnitz, called the differential notation, which we now proceed to explain; not however for the purpose of abandoning the former notation, for we shall often make use of it, as it is preferable in certain cases to any other.

The ratio of the symbols $df(x)$ and dx taken to represent $f'(x)$.

81. In the differential notation instead of representing the limiting ratio of $\frac{f(x')-f(x)}{x'-x}$ when x' approaches x, by a single quantity, such as $f'(x)$, we represent it by the ratio of two arbitrary quantities denoted by the symbols $df(x)$ and dx, which, for a reason we shall explain, are called the *differentials* of $f(x)$ and x; the letter d being simply an abbreviation of the words "*differential of.*"

Definition of $df(x)$ and dx.

We define $df(x)$ and dx as follows:

$\mathrm{d}\,\mathrm{f}(\mathrm{x})$ *and* dx *are two quantities whose ratio* $\frac{\mathrm{d}\,\mathrm{f}(\mathrm{x})}{\mathrm{dx}}$ *is equal to the limiting value of* $\frac{\mathrm{f}(\mathrm{x}')-\mathrm{f}(\mathrm{x})}{\mathrm{x}'-\mathrm{x}}$ *when* x' *approaches* x. *Or more simply;* $\mathrm{d}\,\mathrm{f}(\mathrm{x})$ *and* dx *are two quantities such that* $\frac{\mathrm{d}\,\mathrm{f}(\mathrm{x})}{\mathrm{dx}}=\mathrm{f}'(\mathrm{x})$.

In this definition, all that we say of $df(x)$ and dx is this, "*that they are in a certain ratio;*" therefore they are arbitrary quantities, *and we may give to either of them any value we please, constant or variable*, provided we give to the other that value which makes $\dfrac{df(x)}{dx} = f'(x)$.

One of the quantities $df(x)$, dx is quite arbitrary.

82. It may be asked respecting this notation, is there not some degree of vagueness in representing a definite quantity $f'(x)$ by the ratio of two arbitrary quantities?

Objection to this notation answered.

To this we may answer, that we often do the very same thing in common Algebra without any vagueness; we often say let $\dfrac{m}{n}$ represent such and such a quantity, instead of let m represent such a quantity. Thus, suppose it required to divide a quantity a into two parts in a given ratio: we might proceed thus. Let m and n be any two quantities which are in the given ratio, and let x and y be the two parts, then $x + y = a$, and $\dfrac{x}{y} = \dfrac{m}{n}$;

$$\therefore \left(\frac{m}{n} + 1\right) y = a, \quad \text{or,} \quad y = \frac{na}{m+n}; \quad \text{and} \therefore x = \frac{ma}{m+n}.$$

Or we might proceed thus, let m be the given ratio, then

$$\frac{x}{y} = m, \quad (m+1)\, y = a; \quad \text{and} \therefore y = \frac{a}{m+1}\; x = \frac{ma}{m+1}.$$

It is evident that there is not the least degree of vagueness here in representing the given ratio by $\dfrac{m}{n}$, instead of simply by m*,

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* The chief practical advantages we gain by representing $f'(x)$ by the ratio $\frac{df(x)}{dx}$ seem to me to be these. We may often suppose $df(x)$ or dx to have some value which will simplify our expressions (as will appear in changing the independant variable hereafter.) The application of Rule XIII. to complex functions is very much facilitated by using the notation $\frac{df(x)}{dx}$. We may often advantageously preserve the symmetry of our expressions by using this notation. It is almost

Advantages of the notation $\frac{df(x)}{dx}$.

Origin of the notation $\frac{df(x)}{dx}$, and of the term differential.

83. Again it may be asked, what is the reason of using the letter d before $f(x)$ and x, to represent the terms of the ratio by which we denote $f'(x)$? This question may be answered by the following very brief account of the origin of this notation and of the term differential.

$f(x') - f(x)$, and $x' - x$ are the differences between corresponding values of $f(x)$ and x, and they are often represented by a δ prefixed to $f(x)$ and x, in this manner, viz. $\delta f(x)$, δx; δ being simply an abbreviation of the words "*the difference between two values of.*"

Now, $\delta f(x)$ and δx become zero when $x' = x$, and then their ratio $\frac{\delta f(x)}{\delta x}$ ceases to be a definite quantity; but so long as x' is not actually equal to x the ratio $\frac{\delta f(x)}{\delta x}$ is a definite quantity. We may therefore conceive $\delta f(x)$ and δx as small as we please though not actually zero, without rendering our conception of the ratio $\frac{\delta f(x)}{\delta x}$ at all vague or difficult: indeed it is just as easy to conceive that a definite ratio subsists between $\delta f(x)$ and δx when they are in a state of extreme smallness as when they are of ordinary magnitude for our idea of a ratio is quite independent of the *actual* magnitude of the quantities composing it.

The differences $\delta f(x)$ and δx when in a state of extreme smallness were called *differentials* by Leibnitz (i.e. minute differences), and the symbols $df(x)$ and dx were made use of by him to represent them; d, like δ, being simply an abbreviation of the words "*differential of.*"

In all calculations into which these differentials entered he supposed them to be what are called *infinitesimals*, i.e. quantities so small, that they may without error be neglected compared with ordinary quantities, and on this sup-

impossible to represent what are called total differentials without this notation. It is peculiarly adapted to the case of definite and multiple integrals in the Integral Calculus. And it is a very expressive notation, which makes it peculiarly convenient in mixed mathematics; e.g. in the case of the principle of virtual velocities applied to an example.

position he investigated rules for finding $df(x)$ in all cases, and made various interesting applications of these rules. In this manner it was that the differential calculus came into existence.

Example of the method of Leibnitz.

84. In order to explain more distinctly in what this method of Liebnitz consists, we shall apply it to the example given in (23).

According to Leibnitz, we may conceive the triangle PQO (fig. 6) to become so extremely small, that the line PQ shall coincide in direction with the tangent at P; in which case the sides of the triangle OPQ will become infinitesimal quantities; then OQ or $f(x') - f(x)$ will be represented by $df(x)$ and PO by dx, and we shall have

$$\tan QPO \text{ or } \tan PTX = \frac{QO}{PO} = \frac{df(x)}{dx}.$$

Now let $$f(x) = \frac{x^2}{4m},$$

then $$f(x') - f(x) \text{ or } df(x) = \frac{x'^2 - x^2}{4m} = \frac{(x' + x)(x' - x)}{4m}$$
$$= \frac{(2x + dx)\,dx}{4m},$$

putting $x + dx$ for x'.

Now, according to the supposition that differentials are to be neglected compared with ordinary quantities, we must consider $2x + dx$ to be the same thing as $2x$, and therefore we have

$$df(x) = \frac{x}{2m}dx, \text{ and } \therefore \tan PTX = \frac{x}{2m},$$

which is the same result we obtained before.

Where this method is faulty. Reason why the result is correct.

85. Whether this is a strictly logical way of proceeding may be fairly questioned, though the conclusion arrived at is true: for it is easy to see that the correctness of the result arises from the compensation of two errors, namely, the erroneous supposition that the line PQ is coincident with the tangent at P, which it never can be so long as the

triangle POQ has any existence, and the erroneous supposition that $2x + dx$ is equal to $2x$.

We shall endeavour however to shew hereafter, by means of the principles already established, that this method of using differentials *must* always lead to correct results, certain precautions being taken; and this is important to shew, since this method is extensively used, and indeed *must* be used in many cases to avoid complexity. At present we shall say no more upon this subject, except just to remark, that this is the manner in which the notation $\frac{df(x)}{dx}$ has come to represent the limiting value of $\frac{f(x') - f(x)}{x' - x}$ when x' approaches x: for it is easy to see, by the example just given, that $\frac{df(x)}{dx}$, simplified by neglecting the differentials $df(x)$ and dx compared with ordinary quantities, will always be the same thing as this limiting value.

We adopt the notation $\frac{df(x)}{dx}$ and the term differential, but not the notion that $df(x)$ and dx are infinitesimals.

86. It is important to observe here, that we adopt *only* the *notation* $\frac{df(x)}{dx}$ and the *term Differential.* We do not define $df(x)$ and dx to be what $\delta f(x)$ and δx become when in a state of extreme smallness; all that we say is this, that $df(x)$ and dx are two quantities, be they small or large, whose ratio is equal to $f'(x)$, i.e. the limiting value of $\frac{\delta f(x)}{\delta x}$ or $\frac{f(x') - f(x)}{x' - x}$ when x' approaches x.

Geometrical representation of the quantities $df(x)$ and dx.

87. In the example just considered, if we produce PO to any point U and draw US perpendicular to PU to meet SPT in S; then

$$\frac{SU}{PU} = \tan PTS = \text{lim. val. of} \frac{f(x') - f(x)}{x' - x} \text{ when } x' \text{ app. } x,$$
$$= f'(x);$$

hence SU and PU are the differentials of $f(x)$ and x, since they are two quantities whose ratio $= f'(x)$; we have therefore $SU = df(x)$, $PU = dx$.

88. Since $\frac{df(x)}{dx}$ and $f'(x)$ represent the same thing, we have

How $f'(x)$ came to be called the differential coefficient.

$$\frac{df(x)}{dx} = f'(x), \text{ and } \therefore\ df(x) = f'(x)\,dx,$$

$f'(x)$ is therefore the coefficient by which we must multiply the differential of x to obtain the differential of $f(x)$; on this account $f'(x)$ has acquired the name of "*the differential coefficient*," and it is in this way that $\frac{df(x)}{dx}$, which $=f'(x)$, has come by the same name.

The obtaining of the relation between dy and dx is called *Differentiation*.

89. We may express the Rules Chap. V. in the differential notation by simply putting

The rules obtained in last chapter stated in the differential notation.

$$\frac{df(x)}{dx}, \quad \frac{d\phi(x)}{dx}, \quad \frac{d\psi(x)}{dx}, \text{ \&c....}$$

for $f'(x)$, $\phi'(x)$, $\psi'(x)$, &c. respectively, and multiplying up dx. For the sake of neatness we shall put y, u, v, for $f(x)$, $\phi(x)$, $\psi(x)$, respectively: and therefore dy, du, dv, &c.... for $f'(x)\,dx$, $\phi'(x)\,dx$, $\psi'(x)\,dx$...&c. In this manner we have

Rule I. If $y = \text{constant}$, then $dy = 0$.

Rule II. If $y = cu$, $dy = cdu$.

Rule III. If $y = u + v + w...$, $dy = du + dv + dw...$

Rule IV. If $y = uv$, $dy = vdu + udv$.

If $y = \frac{u}{v}$, $dy = \frac{vdu - udv}{v^2}$.

Rule V. If $y = x^n$, $dy = nx^{n-1}dx$.

Rule VI. If $y = \log x$, $dy = \dfrac{dx}{x}$.

Rule VII. If $y = a^x$, $dy = \log a . a^x dx$.

Rule VIII. If $y = \sin x$, $dy = \cos x dx$.

If $y = \cos x$, $dy = -\sin x dx$.

Rule IX. If $y = \tan x$, $dy = \dfrac{dx}{\cos^2 x}$.

If $y = \cot x$, $dy = -\dfrac{dx}{\sin^2 x}$.

Rule X. If $y = \sin^{-1} x$, $dy = \dfrac{dx}{\sqrt{1-x^2}}$.

If $y = \cos^{-1} x$, $dy = -\dfrac{dx}{\sqrt{1-x^2}}$.

Rule XI. If $y = \tan^{-1} x$, $dy = \dfrac{dx}{1+x^2}$.

If $y = \cot^{-1} x$, $dy = -\dfrac{dx}{1+x^2}$.

Rule XII. 90. **Rule XII.** shews that if we obtain dx in terms of dy from the equation $x = \phi(y)$, the value of $\dfrac{dy}{dx}$ so obtained is the proper value; i. e. the same value that would be obtained if we found y in terms of x directly, and then differentiated. For by differentiating the equation $x = \phi(y)$ as it stands, we find

$$dx = \phi'(y) dy, \quad \text{and } \therefore \frac{dy}{dx} = \frac{1}{\phi'(y)}.$$

Now if $f(x)$ be the value of y found directly in terms of x, we have $\dfrac{dy}{dx} = f'(x)$; which is the same as the former value, since

$f'(x) = \frac{1}{\phi'(y)}$ by Rule XII. Hence, whichever way we proceed, the value of $\frac{dy}{dx}$ obtained is the same.

91. Rule XIII. shews that if we find dy in terms of dv from the equation $y = \phi(v)$, and dv in terms of dx from the equation $v = \psi(x)$, and so dy in terms of dx; the value of $\frac{dy}{dx}$ thus found is the proper value; i. e. the same value that would be obtained if we found y in terms of x directly by eliminating v, and then differentiated. For by differentiating the equations $y = \phi(v)$, $v = \psi(x)$, we obtain

Rule XIII.

$$dy = \phi'(v)\,dv, \quad dv = \psi'(x)\,dx, \quad \text{and } \therefore \frac{dy}{dx} = \phi'(v)\,\psi'(x).$$

Now if $f(x)$ be the value of y found directly in terms of x, we have $\frac{dy}{dx} = f'(x)$, which is the same as the former value, since $f'(x) = \phi'(v)\,\psi'(x)$ by Rule XIII. Hence, whichever way we proceed, the value of $\frac{dy}{dx}$ obtained is the same.

92. If we use the differential notation, this Rule does not appear to want any proof; for it is stated in that notation, as follows. If y be a function of v, and v a function of x, then $\frac{dy}{dx} = \frac{dy}{dv}\,\frac{dv}{dx}$. Now, since dv divides out in the second member, this appears to be self-evident. But we must remember that the $\frac{dy}{dx}$ in the first member is supposed to be derived from y expressed directly in terms of x; whereas the $\frac{dy}{dx}$ obtained from the second member, by dividing out dv, is not derived from y in this manner; and though it is easy to see that the result is the same in both cases, yet it ought to be formally proved for the sake of exactness.

Rules XII and XIII seem to want no proof in the differential notation. Why nevertheless they ought to be proved.

The same may be said of the preceding Rule also.

For examples of the use of the differential notation, and of the Rules for differentiation in general, see Appendix (C).

Rule XV.

†† 93. To express Rule XV. in the differential notation, we shall assume $d_u y$ to denote the differential coefficient of y on the supposition that v is constant and u alone variable, and $d_v y$ to denote the differential coefficient of y on the supposition that u is constant and v alone variable; then we have

$$\phi'(\underset{.}{u}v) = d_u y, \quad \phi'(u\underset{.}{v}) = d_v y, \quad \psi'(x) = \frac{du}{dx}, \quad \chi'(x) = \frac{dv}{dx},$$

$$\text{and} \quad f'(x) = \frac{dy}{dx}, \text{ and therefore}$$

$$\frac{dy}{dx} = d_u y \frac{du}{dx} + d_v y \frac{dv}{dx};$$

$$\text{or} \quad dy = d_u y \, du + d_v y \, dv.$$

Hence Rule XV. may be stated thus.

If y be a function of two other functions of x u and v, then

$$dy = d_u y \, du + d_v y \, dv,$$

Differential of a function equal the sum of its partial differentials.

$d_u y \, du$ and $d_v y \, dv$ may be called the partial differentials of y with respect to u and v respectively. *Hence the differential of* y *is equal to the sum of its partial differentials with respect to* u *and* v *respectively.*

More generally, if y be a function of any number of functions of x, viz. u, v, w..., we have by Cor. to Rule XV.

$$dy = d_u y \, du + d_v y \, dv + d_w y \, dw \ldots\ldots$$

The manner in which we arrive at the conception of the differential of a function

†† 94. Suppose that y is a function of u, v, w,... where u, v, w,... are variables to which we may assign any values we please independently of each other; then we may suppose u, v, w,... to be any arbitrary functions of a new variable x; for by altering the nature of these functions we may make

of several *independant* variables.

$u, v, w,\ldots$ vary in any manner we please when x varies, and so receive any arbitrary values: in fact a function of $u, v, w, \ldots$ where $u, v, w,\ldots$ are any arbitrary functions of x, is just the same thing as a function of $u, v, w,\ldots$ where $u, v, w,\ldots$ are any arbitrary quantities. The advantage of looking upon $u, v, w,\ldots$ as arbitrary functions of x, instead of mere arbitrary quantities, is this, that we thereby arrive very simply at the conception of the differential of a function of several independant variables: for, if y be a function of the independant variables $u, v, w,\ldots$, we may, if we choose, consider $u, v, w, \ldots$ to be arbitrary functions of a new variable x, and then by the result of the previous article we have

$$dy = d_u y\, du + d_v y\, dv + d_w y\, dw + \ldots\ldots$$

Thus if y *be a function of several independant variables, the differential of* y, *regarded in this manner, is the sum of the partial differentials of* y *with respect to each of the variables.*

du, dv, dw, generally supposed to be arbitrary constants.

What the quantities $du, dv, dw\ldots$ are depends upon what functions we suppose $u, v, w\ldots$ to be of x, and what value we give to dx; if we suppose $u = \phi(x)$, $v = \psi(x)$, $w = \chi(x)$, then $du = \phi'(x)\,dx$, $dv = \psi'(x)\,dx$, $dw = \chi'(x)\,dx\ldots$; and hence since ϕ, ψ, χ may be any functions whatever, and dx any quantity, it is clear that $du, dv, dw\ldots$ may have any values whatever we please to give them. We shall in general suppose that $u = Ax$, $v = Bx$, $w = Cx\ldots$ where $A, B, C\ldots$ are any arbitrary constants, and that dx is constant; and then $du, dv, dw\ldots$ will be arbitrary constants.

Definition of the differential of a function of several independant variables.

We shall therefore assume that if y be a function of several independant variables $u, v, w\ldots$ then the differential of y, which we shall denote by dy, is equal to the expression

$$d_u y\,.\,du + d_v y\,.\,dv + d_w y\,.\,dw\ldots$$

where du, dv, dw, &c. are in general any arbitrary constants; but may also receive any variable values we please to give them.

This assumption is really our definition of a differential of several *independant* variables; which definition the peculiar form assumed by the differential of a function of several *dependant* variables suggests to us in the manner we have explained.

A total differential; what.

dy here is often called the *Total Differential* of y, in contradistinction to the partial differentials $d_u y\, du$, $d_v y\, du$, &c.

Examples.

Let $y = u^3 + v^3 + w^3 - 3uvw$,

$$\text{then}\quad d_u y = 3(u^2 - vw),$$
$$d_v y = 3(v^2 - uw)$$
$$d_w y = 3(w^2 - uv);$$

$$\text{and} \ \therefore\ dy = 3\{(u^2 - vw)\,du + (v^2 - uw)\,dv + (w^2 - uv)\,dw\}.$$

Let $y = u^{vw}$,

$$\text{then}\quad dy = u^{vw-1}\{vw\,du + u\log u\,(v\,dw + w\,dv)\}.$$

Rule XVI.

†† 95. Rule XVI. is thus expressed in the differential notation.

If U be a function of x and y, and we put $U = 0$, then

$$d_x U \,.\, dx + d_y U \,.\, dy = 0,$$

from which equation we immediately get $\dfrac{dy}{dx}$.

Example.

Let the given equation be

$$U = \sin x \sin y - xy = 0,$$

$$\text{here}\ d_x U = (\cos x \sin y - y), \quad d_y U = (\sin x \cos y - x);$$

$$\therefore\ (\cos x \sin y - y)\,dx + (\sin x \cos y - x)\,dy = 0,$$

$$\text{and}\ \therefore\ \frac{dy}{dx} = -\frac{\cos x \sin y - y}{\sin x \cos y - x}.$$

CHAPTER VII.

SUCCESSIVE DERIVATIVES OR DIFFERENTIALS. CHANGE OF THE INDEPENDANT VARIABLE.

96. In the same manner that we obtain the derivative $f'(x)$ of any function $f(x)$, so we may obtain the derivative of $f'(x)$, and again the derivative of that derivative, and so on. Thus if $f(x) = x^n$, $f'(x) = nx^{n-1}$, and the derivative of this is $n(n-1)x^{n-2}$; again the derivative of $n(n-1)x^{n-2}$ is $n(n-1)(n-2)x^{n-3}$, and so we may go on.

The n^{th} derivative or differential coefficient; what. We represent it by $f^n(x)$.

The derivative of $f'(x)$ is called the *Second Derivative* of $f(x)$, and is denoted by $f^2(x)$; the derivative of $f^2(x)$ is called the *Third Derivative* of $f(x)$, and is denoted by $f^3(x)$, and so on; and in general $f^n(x)$ denotes the

deriv. of [the deriv. of {the deriv.... (to n derivs.)}] of $f(x)$,

$f^n(x)$ is also called the n^{th} differential coefficient of $f(x)$.

$f^n(x)$ is represented by $\frac{d^n y}{dx^n}$ in the differential notation; on what hypothesis.

97. If we put $f(x) = y$, and therefore $f'(x) = \frac{dy}{dx}$, then

$$f^2(x) = \frac{df'(x)}{dx} = \frac{d\left(\frac{dy}{dx}\right)}{dx}.$$

Now we have seen in (80) that we may give to one of the quantities dy or dx any value we please, variable or constant; let us suppose dx a constant, then by Rule II,

$$d\left(\frac{dy}{dx}\right) = \frac{ddy}{dx}, \text{ and we have therefore } f^2(x) = \frac{ddy}{dx^2}.$$

$$\text{Again } f^3(x) = \frac{df^2(x)}{dx} = \frac{d\left(\frac{ddy}{dx^2}\right)}{dx} = \frac{dddy}{dx^3},$$

as before, and so on, and in general

$$f^n(x) = \frac{ddd\ldots(\text{to } n\ d\text{'s})\, y}{dx^n}.$$

To avoid the repetition of d's we represent d repeated n times by d^n, and thus we have

$$f^n(x) = \frac{d^n y}{dx^n}.$$

For examples of finding successive differential coefficients, see Appendix (D).

Expression for $f^n(x)$ when dx is not supposed constant.

98. We have supposed dx constant for the sake of simplicity; if we do not make this supposition, then we have

$$f^2(x) = \frac{d\left(\frac{dy}{dx}\right)}{dx},\quad f^3(x) = \frac{df^2(x)}{dx} = \frac{d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}}{dx} \ldots \&c. \ldots \&c.$$

which are the expressions for the successive derivatives of y considered as a function of x, whatever be the values of dx and dy. If we perform the differentiations, and put d^2 for dd, d^3 for ddd, &c., &c., we have

$$d\left(\frac{dy}{dx}\right) = \frac{d^2y\,dx - dy\,d^2x}{dx^2}, \text{ and } \therefore f^2(x) = \frac{d^2y\,dx - dy\,d^2x}{dx^3},$$

$$df^2(x) = \frac{(d^3y\,dx - dy\,d^3x)\,dx^3 - (d^2y\,dx - dy\,d^2x)\,3\,dx^2\,d^2x}{dx^6};$$

$$\therefore f^3(x) = \frac{(d^3y\,dx - dy\,d^3x)\,dx - 3\,(d^2y\,dx - dy\,d^2x)\,d^2x}{dx^5};$$

and so we may go on. The expressions however become extremely complicated.

In these expressions, if we suppose dx constant, we have

$$d^2x = 0,\quad d^3x = 0,\ \&c. \ldots \text{ and therefore}$$

$$f^2(x) = \frac{d^2y}{dx^2},\quad f^3(x) = \frac{d^3y}{dx^3},\ \&c.,$$

which are of course the same expressions as in the preceding article.

If we suppose dy constant, we have $d^2y = 0$, $d^3y = 0$, and therefore

$$f^2(x) = -\frac{dy\,d^2x}{dx^3}, \quad f^3(x) = -\frac{dy\,\{d^3x\,dx - 3\,(d^2x)^2\}}{dx^5} \ldots$$

and so on.

Independant variable; what.

99. That variable whose differential is supposed to be constant is called the independant variable; thus when we put $f^2(x) = \frac{d^2y}{dx^2}$, x is the independant variable; when we put $f^2(x) = -\frac{dy\,d^2x}{dx^3}$, y is the independant variable; and when we put $f^2(x) = \frac{d^2y\,dx - dy\,d^2x}{dx^3}$, neither x nor y is the independant variable.

What d^ny represents depends upon what we consider to be the independant variable.

100. It is important to remember that $\frac{d^ny}{dx^n}$ represents $f^n(x)$, *only* on the supposition that x is the independant variable, and therefore, though we may always put $d^ny = f^n(x)\,dx^n$ when x is supposed to be the independant variable, we may not do so if this be not supposed. d^ny, therefore, does not always represent the same quantity; what it represents depends upon what quantity is considered the independant variable.

Example.

Thus suppose $y = x^2$, then $dy = 2x\,dx$, and $d^2y = 2\,dx^2$, if dx be constant. But suppose that $x = z^2$, and $\therefore\ y = z^4$; then $dy = 4z^3dz$, $d^2y = 12z^2dz^2$, if dz be constant; now since $x = z^2$, $dx = 2z\,dz$, and $\therefore$ $12z^2dz^2 = 3\,dx^2$, and $\therefore$ $d^2y = 3\,dx^2$. Hence, when x is supposed the independant variable, $d^2y = 2\,dx^2$, and when z or $x^{\frac{1}{2}}$ is supposed the independant variable, $d^2y = 3\,dx^2$: from which it is clear that d^2y does not mean the same thing when x is the independant variable, and when $x^{\frac{1}{2}}$ is.

101. It is often necessary to change the independant variable in an expression containing differentials; this we may

Change of the independant variable; how effected.

easily do by remembering always, 1st, that the independant variable is that variable whose differential is constant; 2ndly, that $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c. represent the successive derivatives of y considered as a function of x, on the express condition that dx is a constant; and 3rdly, that these derivatives, whatever be the independant variable, are represented by

$$\frac{dy}{dx},\quad \frac{d\left(\frac{dy}{dx}\right)}{dx},\quad \frac{d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}}{dx} \ldots \text{\&c. \&c.} \ldots \text{ respectively.}$$

Hence, if we have any expression involving

$$\frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \frac{d^3y}{dx^3} \ldots \text{\&c.}$$

where these quantities are supposed to represent the successive derivatives of y considered as a function of x, on which supposition x must necessarily be the independant variable: then since

$$\frac{dy}{dx},\quad \frac{d\left(\frac{dy}{dx}\right)}{dx},\quad \frac{d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}}{dx} \ldots \text{\&c.}$$

represent the same derivatives, we may put these instead of

$$\frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \frac{d^3y}{dx^3} \ldots \text{\&c. respectively;}$$

and then we may suppose any quantity we please to be the independant variable, since the substituted expressions represent the successive derivatives of y, whatever be the independant variable. If the expression involve dx, dy, d^2y, d^3y, &c. but not in the precise forms

$$\frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \frac{d^3y}{dx^3},\ \text{\&c.}$$

then we substitute in this manner

$$\frac{d^2y}{dx^2} = \frac{d\left(\dfrac{dy}{dx}\right)}{dx}; \quad \therefore\ d^2y = dx\, d\left(\frac{dy}{dx}\right),$$

which we put for d^2y,

$$\frac{d^3y}{dx^3} = \frac{d\left\{\dfrac{d\left(\dfrac{dy}{dx}\right)}{dx}\right\}}{dx}; \quad \therefore\ d^3y = dx^2\, d\left\{\frac{d\left(\dfrac{dy}{dx}\right)}{dx}\right\},$$

which put for d^3y, and so on.

In this manner we may change the independant variable in any expression. The following examples will make this clear.

102. In the expression Example.

$$\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = u,$$

where x is the independant variable, to make θ the independant variable, $\cos\theta$ being equal to x.

Making the necessary substitutions for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, we have

$$u = \frac{d\left(\dfrac{dy}{dx}\right)}{dx} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2}.$$

In this expression we are at liberty to assume any quantity we please to be the independant variable; let us suppose therefore that θ is, and then we have

$$dx = -\sin\theta\, d\theta, \quad \frac{dy}{dx} = -\frac{dy}{d\theta\sin\theta};$$

$$\therefore\ d\left(\frac{dy}{dx}\right) = -\frac{d^2y\sin\theta - dy\cos\theta\, d\theta}{d\theta\sin^2\theta},$$

since $d\theta$ is now constant.

Hence $$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y \sin\theta - dy \cos\theta \, d\theta}{d\theta \sin^2\theta}$$

$$-\frac{x}{1-x^2}\,\frac{dy}{dx} = \frac{\cos\theta}{\sin^2\theta}\,\frac{dy}{d\theta \sin\theta},$$

$$\frac{y}{1-x^2} = \frac{y}{\sin^2\theta}.$$

Hence $$u = \left(\frac{d^2y}{d\theta^2} + y\right)\frac{1}{\sin^2\theta},$$

in which expression θ is supposed to be the independant variable.

Another Example.

103. Given the expression

$$\left(3\frac{d^2y}{dy^2} - 1\right)\frac{d^2y}{dy^2} - \frac{d^3y}{dy^3} = u,$$

in which x is the independant variable; to find what it becomes when y is the independant variable.

Here put $$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx}, \text{ and } \therefore\ d^2y = dx\, d\left(\frac{dy}{dx}\right);$$

also put $$\frac{d^3y}{dx^3} = \frac{d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}}{dx},$$

and $$\therefore\ d^3y = dx^2 d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}.$$

and then we have

$$u = \left\{3\frac{dx}{dy^2}\,d\left(\frac{dy}{dx}\right) - 1\right\}\frac{dx}{dy^2}\,d\left(\frac{dy}{dx}\right) - \frac{dx^2}{dy^3}\,d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\}.$$

In this expression we are at liberty to assume any quantity we please to be the independant variable, let us therefore suppose that y is, and then we have

$$d\left(\frac{dy}{dx}\right) = -\frac{dy\,d^2x}{dx^2},$$ since dy is now constant;

$$d\left\{\frac{d\left(\frac{dy}{dx}\right)}{dx}\right\} = -\,dy\left(\frac{d^3x}{dx^3} - \frac{3\,d^2x\,d^2x}{dx^4}\right).$$

Hence $$u = \left(3\frac{d^2x}{dx\,dy} + 1\right)\frac{d^2x}{dx\,dy} + \frac{d^3x}{dx\,dy^2} - 3\,\frac{(d^2x)^2}{dx^2dy^2}$$

$$= \frac{d^2x}{dxdy} + \frac{d^3x}{dxdy^2} = \frac{dy}{dx}\left(\frac{d^2x}{dy^2} + \frac{d^3x}{dy^3}\right),$$

in which expression y is the independant variable.

(For other examples see Appendix E).

The following articles have reference to the successive differentiation of functions of several variables.

Successive differentiation of $f(uv)$, $u\,v$ being functions of x.

†† 104. Suppose that y is some function of u and v, u and v being functions of x; then by Rule XV.

$$dy = d_u y\,du + d_v y\,dv\,;$$

therefore differentiating again

$$d^2y = d\,(d_u\,y\,du) + d\,(d_v\,y\,dv).$$

Now $d\,(d_u\,y\,du) = d\,(d_u y)\,.\,du + d_u\,y\,.\,d^2u$

$$= \{d_u\,(d_u\,y)\,.\,du + d_v\,(d_u\,y)\,.\,dv\}\,du + d_u\,y\,d^2u,$$ by Rule XV.

hence assuming d_u^2y to denote d_ud_uy, we have

$$d\,(d_u\,y\,du) = d_u^2y\,.\,du^2 + d_vd_u\,y\,.\,dudv + d_uyd^2u\,\ldots\,(1)\,;$$

and similarly we have

$$d\,(d_vy\,dv) = d_v^2y\,.\,dv^2 + d_ud_v\,y\,.\,dudv + d_vyd^2v\,\ldots\,(2)\,;$$

and adding (1) and (2), we obtain an expression for d^2y. This expression becomes somewhat simplified by means of the following article.

Proposition $d_u d_v y = d_v d_u y$.

†† 105. *To shew that* $\mathrm{d_u d_v y = d_v d_u y}$.

Let $y = \phi(uv)$; then, by the definition of a partial derivative, if we put

$$\frac{\phi(u'v) - \phi(uv)}{u' - u} \sim d_u y = \psi(u'v),$$

$\psi(u'v)$ may be diminished *ad libitum* by sufficiently diminishing $u' - u$, without altering v; and this is true for all values of v: hence we may evidently diminish $\dfrac{\psi(u'v') - \psi(u'v)}{v' - v}$ *ad libitum*, by sufficiently diminishing $u' - u$ without altering v' or v; and this is true for all values of v' and v (except of course v' actually $= v$). Hence, since we may make $\dfrac{\psi(u'v') - \psi(uv)}{v' - v}$ differ as little as we please from $d_v \psi(u'v)$ by taking $v' - v$ small enough, it clearly must be possible to diminish $d_v \psi(u'v)$ *ad libitum*, by sufficiently diminishing $u' - u$; but

$$d_v \psi(u'v) = \frac{d_v \phi(u'v) - d_v \phi(uv)}{u' - u} \sim d_v d_u y;$$

hence, by Lemma VI, Cor. 2, $d_v d_u y$ must be the limiting value of $\dfrac{d_v \phi(u'v) - d_v \phi(uv)}{u' - u}$ when u' approaches u; but, by the definition of a partial derivative, this limiting value is represented by $d_u \{d_v \phi(uv)\}$, or $d_u d_v y$. Hence we have

$$d_u d_v y = d_v d_u y.$$

It appears therefore that whenever we have $d_u d_v$ written before any expression, we may write $d_v d_u$ instead of it, and *vice versâ:* i.e. the order in which we perform partial differentiations is indifferent.

Cor. $d_u{}^m d_v{}^n . y = d_v{}^n d_u{}^m . y$.

106. Differentiating the result $d_u d_v y = d_v d_u y$ with respect to u, we have

$$d_u^2 d_v y = d_u d_v d_u y = d_v d_u^2 y, \quad \text{putting } d_v d_u \text{ for } d_u d_v;$$

and differentiating this result, $d_u^3 d_v y = d_u d_v d_u^2 y = d_v d_u^3 y$ similarly, and so on, and in general

$$d_u^m d_v y = d_v d_u^m y.$$

It appears therefore that whenever we have $d_u^m d_v$ written before any expression, we may write $d_v d_u^m$ instead of it, and *vice versâ*.

Differentiating the result $d_v d_u^m y = d_u^m d_v y$ with respect to v, we have

$$d_v^2 d_u^m y = d_v d_u^m d_v y = d_u^m d_v^2 y, \text{ putting } d_u^m d_v \text{ instead of } d_v d_u^m,$$

$$\text{and again } d_v^3 d_u^m y = d_v d_u^m d_v^2 y = d_u^m d_v^3 y \text{ similarly,}$$

and so on, and in general

$$d_v^n d_u^m y = d_u^m d_v^n y.$$

It appears therefore that whenever we have $d_v^n d_u^m$ written before any expression, we may write $d_u^m d_v^n$ instead of it.

(Another proof of this result will be given when we come to speak of series in Chapter IX.)

This result shews that when we successively differentiate an expression with respect to different variables, it is no matter in what order these differentiations are performed.

Successive differentiation of $f(uv)$.

†† 107. We shall now return to (104).

d^2y is found by adding (1) and (2); hence, from what has been just proved, we have

$$d^2y = d_u^2 y \,.\, du^2 + 2 d_u d_v y \,.\, du dv + d_v^2 y \,.\, dv^2 + d_u y \,.\, d^2 u + d_v y \,.\, d^2 v.$$

If we differentiate this result again in a similar manner, we obtain

$$d^3 y = d_u^3 y \,.\, du^3 + 3\, d_u^2 d_v y \,.\, du^2 dv + 3\, d_u^2 d_v y \,.\, du dv^2 + d_v^3 y \,.\, dv^3,$$

$$3\, d_u^2 y \,.\, du d^2 u + 3\, d_u d_v y \,.\, (d^2 u dv + du d^2 v) + 3\, d_v^2 y \,.\, dv d^2 v$$

$$+ d_u y \,.\, d^3 u + d_v y \,.\, dv^3\,;$$

and so we may find $d^4 y$, &c.; but the results become extremely complicated, and we shall not put them down.

Example.

Let $y = u^2 + v^2 + uv$,

then $d_u y = 2u + v, \quad d_u^2 y = 2, \quad d_v y = 2v + u, \quad d_v^2 y = 2,$

$d_u d_v y = 1$; and therefore we have

$$d^2y = 2du^2 + 2du\,dv + 2dv^2 + (2u + v)\,d^2u + (2v + u)\,d^2v.$$

Again $d_u^3 y = 0$, $d_v^3 y = 0$, $d_u^2 d_v y = 0$, $d_u d_v^2 y = 0$;

and therefore we have

$$d^3y = 3(2du\,d^2u + d^2u\,dv + dv\,d^2u + 2dv\,d^2v)$$
$$+ (2u + v)\,d^3u + (2v + u)\,d^3v,$$

and so we may find d^4y, &c.

The method here given of finding the successive differentials of a function of several variables is useful only for general purposes. In particular cases we may always find the successive differentials more readily by the common rules of differentiation. (See Appendix F where examples of this are given).

Successive differentiation of an equation between y and x.

†† **108.** If U be a function of x and y, and we put $U = 0$, then we have by Rule XVI.

$$d_x U\,.\,dx + d_y U\,.\,dy = 0,$$

and differentiating as in (107) we have, supposing dx constant,

$$d_x^2 U\,.\,dx^2 + 2d_x d_y U\,.\,dx\,dy + d_y^2 U\,.\,dy^2 + d_y U\,.\,d^2y = 0,$$

and so we may go on, and by this means find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ ... &c. from the equation $U = 0$.

Ex. Let $x^3 - 3axy + y^3 = U = 0$,

then $d_x U = 3x^2 - 3ay$, $d_x^2 U = 6x$, $d_y U = 3y^3 - 3ax$,

$d_y^2 U = 6y$, $d_x d_y U = -3a$: we have therefore

$$(x^2 - ay)\,dx + (y^2 - ax)\,dy = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$2x\,dx^2 - 2a\,dx\,dy + 2y\,dy^2 + (y^2 - ax)\,d^2y = 0 \ldots (2),$$

from (1) we get $\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax}$, and then from (2)

$$2x + 2a\,\frac{x^2 - ay}{y^2 - ax} + 2y\,\frac{(x^2 - ay)^2}{(y^2 - ax)^2} + (y^2 - ax)\,\frac{d^2y}{dx^2} = 0,$$

which gives $\frac{d^2y}{dx^2}$; and so we may find $\frac{d^3y}{dx^2}$, &c.

(See Appendix F.)

Successive differentiation of a function of two independant variables.

†† 109. If y be a function of u and v, u and v being independant variables, then supposing du and dv constant (see 94), we have, as in (107)

$$dy = d_u y \,.\, du + d_v y \,.\, dv,$$

$$d^2y = d_u^2 y \,.\, du^2 + 2\,d_u d_v y \,.\, du\, dv + d_v^2 y \,.\, dv^2,$$

$$d^3y = d_u^3 y \,.\, du^3 + 3\,d_u^2 d_v y \,.\, du^2\, dv + 3\,d_u d_v^2 y \,.\, du\, dv^2 + d_u^3 y \,.\, dv^3,$$

&c. .... &c. ....

We perceive that the coefficients of the terms in the second members of these equations are respectively those in the expansions of $1 + z$, $(1 + z^2)$, $(1 + z)^3 \ldots$

Let 1, A, B, $C \ldots$ be the coefficients in the expansion of $(1 + z)^n$, and let us assume that

$$d^n y = d_u^n y \,.\, du^n + A\,d_u^{n-1} d_v y \,.\, du^{n-1} dv + B\,d_u^{n-2} d_v^2 y \,.\, du^{n-2} dv^2 + \ldots$$

$$\text{then } d^{n+1}y = d_u(d^n y)\,.\,du + d_v(d^n y)\,.\,dv$$

$$= d_u^{n+1} y \,.\, du^{n+1} + \begin{array}{l|}A\\ +1\end{array}\, d_u^n d_v y \,.\, du^n dv + \begin{array}{l|}B\\ +A\end{array}\, d_u^{n-1} d_v^2 y \,.\, du^{n-1} dv^2 + \begin{array}{l}\ldots\\ +\ldots\end{array}$$

Now 1, $A + 1$, $B + A$, &c. we know to be the coefficients in $(1 + z)^{n+1}$ expanded: hence it appears that if the law we have assumed be true for n it is true also for $n + 1$; but we know it to be true for 1, 2, 3; therefore it is true in general. We have therefore

$$d^n y = d_u^n y \,.\, du^n + \frac{n}{1}\, d_u^{n-1} d_v y \,.\, du^{n-1} dv$$

$$+ \frac{n\,(n-1)}{1\,.\,2}\,.\, d_u^{n-2} d^2 y \,.\, du^{n-2} dv^2 + \ldots$$

Symbolical expression for $d^n y$.

††110. This result may be briefly expressed thus,

$$d^n y = (d_u \,.\, du + d_v \,.\, dv)^n . y.$$

By which formula, we do not mean that $d_u \,.\, du$ and $d_v \,.\, dv$ are actually quantities whose sum is raised to the n^{th} power and multiplied into y; but simply this, that if the symbols $d_u \,.\, du$ and $d_v \,.\, dv$ be connected together by the sign $+$, and raised to the n^{th} power in the same manner as if they were ordinary quantities, and if y be written after each term of the result; then the expression so obtained is the proper expression for $d^n y$.

Expression for $d^n y$, when y is a function of more than two independant variables.

††111. We may prove in the same manner that if y be a function of several independant variables u, v, w...&c., then

$$d^n y = (d_u \,.\, du + d_v \,.\, dv + d_w \,.\, dw \ldots)^n . y.$$

CHAPTER VIII.

CERTAIN LEMMAS UPON WHICH THE APPLICATION OF THE DIFFERENTIAL CALCULUS IN MANY CASES DEPENDS.

112. HAVING now concluded what may be considered the first part of the differential calculus, namely that in which we determine a set of rules whereby differentiation may be performed in all cases with facility; we proceed to the second part, in which we shall shew some of the principal uses which may be made of the differential calculus in pure mathematics. But we must previously prove the following lemmas upon which the application of the differential calculus in a great measure depends.

Lemma XVII.

113. *If* $f(x)$ *changes its sign when* x *passes** *through the value* a, *then* $f(a)$ *must be* 0 *or* ∞.

For $f(a)$ is the limiting value of $f(x)$ when x approaches a (Lemma III), and therefore by Lemma IV $f(x)$ has the same sign as $f(a)$ for all values of x taken sufficiently near a; hence, if $f(a)$ be positive, $f(x)$ is positive for all values of x taken sufficiently near a, and therefore cannot change its sign when x passes through the value a; and the same is true if $f(a)$ be negative. If therefore $f(x)$ changes its sign when x passes through the value a, $f(a)$ can neither be positive nor negative; i.e. it must be 0 or $\frac{1}{0}$. Q. E. D.

$f(x)$ does not necessarily change its sign when $f(a)$ is 0 ∞.

114. It is important to observe, that what we have proved here is, not that $f(x)$ *must* change its sign when $f(a)$ is 0 or ∞, but that if *it does change* its sign, $f(a)$ *must* be 0 or ∞.

---

* By saying that, "$f(x)$ changes its sign when x passes through the value a," we mean that when we give x any value a little greater than a, $f(x)$ has a different sign to what it has when we give x any value a little less than a.

Thus, if $f(x) = (x - a)^3$, $f(x)$ changes its sign when x passes through the value a, and here $f(a) = 0$. If $f(x) = \frac{1}{(x-a)^3}$, $f(x)$ changes its sign when x passes through a, and here $f(a) = \infty$. If $f(x) = (x - a)^2$, $f(a) = 0$, but $f(x)$ does not change its sign; and if $f(x) = \frac{1}{(x-a)^2}$, $f(a) = \infty$, but $f(x)$ does not change its sign.

Lemma XVIII.

115. *If we suppose* x *to increase continually,* f(x) *is increasing as long as* f′(x) *continues positive, and is diminishing as long as* f′(x) *continues negative.*

For, by Lemma IV, $\frac{f(x') - f(x)}{x' - x}$ has the same sign as $f'(x)$, for all values of x' taken sufficiently near x: therefore if $f'(x)$ be positive, $f(x') - f(x)$ has the same sign as $x' - x$, and therefore if x' be $> x$, $f(x')$ is $> f(x)$; i. e. $f(x)$ increases when x increases. And similarly, if $f'(x)$ be negative, $f(x') - f(x)$ has the opposite sign to that of $x' - x$, and therefore $f(x')$ is $< f(x)$ if x' be $> x$; i. e. $f(x)$ diminishes when x increases. Hence the truth of the lemma is manifest.

$f'(x)$ measures the rate of variation of $f(x)$ as compared with x.

116. Since $\frac{f(x') - f(x)}{x' - x}$ is very nearly the same thing as $f'(x)$ for all values of x' taken sufficiently near x, it is evident that $f'(x)$ is very nearly the ratio of any small change in $f(x)$ to the corresponding change in x; and therefore the greater $f'(x)$ is, the greater will be the *rate*, so to speak, at which $f(x)$ varies as compared with x.

Hence $f'(x)$, by its sign shews whether $f(x)$ is increasing or diminishing, and by its magnitude the rate at which that increase or diminution takes place.

If we suppose x diminish continually, it is evident that $f(x)$ diminishes or increases according as $f'(x)$ is positive or negative.

117. *If* f (x) *and* ϕ(x) *be any two functions of* x, *then* $\frac{f'(x)}{\phi'(x)}$ *is in general the limiting value of* $\frac{f(x')-f(x)}{\phi(x')-\phi(x)}$ *when* x′ *approaches* x. Lemma XIX.

$$\text{For } \frac{f(x')-f(x)}{\phi(x')-\phi(x)} = \frac{\dfrac{f(x')-f(x)}{x'-x}}{\dfrac{\phi(x')-\phi(x)}{x'-x}}$$

and the limiting value of the second member, by Lemma VIII, is evidently $\frac{f'(x)}{\phi'(x)}$ in general. Q. E. D.

We say "*in general,*" because $\frac{f'(x)}{\phi'(x)}$ may become illusory for some particular value of x, and then of course this Lemma fails. In such a case the following Lemma will take its place.

118. *If* a *be any value of* x *uhich makes* $\frac{f'(x)}{\phi'(x)}$ *illusory, then the limiting value of* $\frac{f'(x)}{\phi'(x)}$ *when* x *approaches* a *is also the limiting value of* $\frac{f(x)-f(a)}{\phi(x)-\phi(a)}$ *when* x *approaches* a. Lemma XX.

By the last proposition $\frac{f'(x)}{\phi'(x)}$ is in general the limiting value $\frac{f(x')-f(x)}{\phi(x')-\phi(x)}$ when x' approaches x; therefore

$$\frac{f(x')-f(x)}{\phi(x')-\phi(x)} \sim \frac{f'(x)}{\phi'(x)}$$

may be diminished *ad libitum* by sufficiently diminishing $x' \sim x$; and this is true for all values of x which do not make $\frac{f'(x)}{\phi'(x)}$ illusory; therefore it is true if we put $x' = a$ and sufficiently diminish $x' \sim a$; i. e. we may diminish

$$\frac{f(a) - f(x)}{\phi(a) - \phi(x)} \sim \frac{f'(x)}{\phi'(x)}$$

ad libitum, by sufficiently diminishing $x \sim a$; hence by Lemma VI, Cor. 2, the limiting value of $\frac{f'(x)}{\phi'(x)}$ is also the limiting value of $\frac{f(a) - f(x)}{\phi(a) - \phi(x)}$ or $\frac{f(x) - f(a)}{\phi(x) - \phi(a)}$ when x approaches a. Q. E. D.

Cor. 1. 119. *If* a *be a value of* x *which makes the quantities* $\mathrm{f}^1(\mathrm{a})$, $\mathrm{f}^2(\mathrm{a}) \ldots \mathrm{f}^{n-1}(\mathrm{a})$, $\phi^1(\mathrm{a})$, $\phi^2(\mathrm{a}) \ldots \phi^{n-1}(\mathrm{a})$, *each zero, and if* $\frac{\mathrm{f}^n(\mathrm{a})}{\phi^n(\mathrm{a})}$ *be not illusory; then* $\frac{\mathrm{f}^n(\mathrm{a})}{\phi^n(\mathrm{a})}$ *is the limiting value of* $\frac{\mathrm{f}(\mathrm{x}) - \mathrm{f}(\mathrm{a})}{\phi(\mathrm{x}) - \phi(\mathrm{a})}$ *when* x *approaches* a.

For, by the Lemma, when x approaches a,

$$\text{lim. val. of } \frac{f(x) - f(a)}{\phi(x) - \phi(a)} = \text{lim. val. of } \frac{f'(x)}{\phi'(x)},$$

$$\text{or of } \frac{f'(x) - f'(a)}{\phi'(x) - \phi'(a)},$$

since $f'(a)$ and $\phi'(a)$ are each zero,

$$= \text{lim. val. of } \frac{f^2(x)}{\phi^2(x)} \text{ by the Lemma,}$$

$$\text{or of } \frac{f^2(x) - f^2(a)}{\phi^2(x) - \phi^2(a)},$$

since $f^2(a)$ and $\phi^2(a)$ are each zero;

and so on, till we come to

$$\text{lim. val. of } \frac{f^{n-1}(x) - f^{n-1}(a)}{\phi^{n-1}(x) - \phi^{n-1}(a)},$$

$$\text{which } = \frac{f^n(a)}{\phi^n(a)}, \text{ by Lemma XIX.}$$

Hence $\dfrac{f^n(a)}{\phi^n(a)}$ is the limiting value of $\dfrac{f(x) - f(a)}{\phi(x) - \phi(a)}$ when x approaches a. Q. E. D.

120. *If* a *be a value of* x *which makes* $f^1(a)$, $f^2(a)$ Cor. 2. $\ldots f^{n-1}(a)$ *each zero, then* $\dfrac{f^n(a)}{\Gamma n}$ *is the limiting value of* $\dfrac{f(x) - f(a)}{(x-a)^n}$ *when* x *approaches* a.

This is easily proved by putting $\phi(x) = (x-a)^n$ in the last Cor., and therefore $\phi(x) - \phi(a) = (x-a)^n$, $\phi^1(a) = 0$, $\phi^2(a) = 0 \ldots \phi^{n-1}(a) = 0$, $\phi^n(a) = \Gamma n$; which evidently gives us $\dfrac{f^n(a)}{\Gamma n}$ for the limiting value of $\dfrac{f(x) - f(a)}{(x-a)^n}$ when x approaches a. Q. E. D.

Conversely: *If a finite quantity* A *be the limiting value* Cor. 3. *of* $\dfrac{f(x) - f(a)}{(x-a)^n}$ *when* x *approaches* a, *then must* $f^1(a) = 0$, $f^2(a) = 0$, $f^3(a) = 0 \ldots f^{n-1}(a) = 0$, *and* $f^n(a) = \Gamma n . A$.

For if we have $f^1(a) = 0$, $f^2(a) = 0 \ldots f^{r-1}(a) = 0$, but $f^r(a)$ not $= 0$, r being any integer less than n; then the limiting value of $\dfrac{f(x) - f(a)}{(x-a)^r}$ when x approaches a will be $\dfrac{f^r(a)}{\Gamma r}$, and therefore that of $\dfrac{f(x) - f(a)}{(x-a)^n}$, which

$$= \frac{1}{(x-a)^{n-r}} \cdot \frac{f(x) - f(a)}{(x-a)^r},$$

will be $\dfrac{1}{0} \cdot \dfrac{f^r(a)}{\Gamma r}$ (Lemma VIII.); which is infinite, contrary to hypothesis. Hence the truth of the Cor. is evident.

121. *If* $f^1(a)$, $f^2(a) \ldots f^{n-1}(a)$, *be each zero, and* Cor. 4. $f^n(a)$ *not zero, then, for all values of* x *taken sufficiently near* a, $f(x) - f(a)$ *has the same sign as* $f^n(a)(x-a)^n$.

For, by Lemma IV, $\dfrac{f(x) - f(a)}{(x-a)^n}$ has the same sign as its limit $\dfrac{f^n(a)}{\Gamma n}$ for all values of x sufficiently near a, and therefore $f(x) - f(a)$ has the same sign as $f^n(a)(x-a)^n$. Q. E. D.

Lemma XXI.

122. *If* f(x) *be any function of* x *we may assume in general that*

$$\mathrm{f}(\mathrm{x}) = \mathrm{f}(\mathrm{a}) + \mathrm{f}'(\mathrm{a})\frac{\mathrm{x}-\mathrm{a}}{\Gamma 1} + \mathrm{f}^2(\mathrm{a})\frac{(\mathrm{x}-\mathrm{a})^2}{\Gamma 2} + \mathrm{f}^3(\mathrm{a})\frac{(\mathrm{x}-\mathrm{a})^3}{\Gamma 3} \;\ldots\ldots$$

$$+ \left\{\mathrm{f}^{\mathrm{n}}(\mathrm{a}) + \mathrm{Q}\right\}\frac{(\mathrm{x}-\mathrm{a})^{\mathrm{n}}}{\Gamma \mathrm{n}},$$

where Q *is some function of* x *which may be diminished* ad libitum *by sufficiently diminishing* x ~ a.

For assume $F(x)$ to represent the quantity

$$f(x) - \left\{f(a) + f'(a)\frac{x-a}{\Gamma 1} + f^2(a)\frac{(x-a)^2}{\Gamma 2} \ldots f^n(a)\frac{(x-a)^n}{\Gamma n}\right\} \ldots (1),$$

then, by differentiating this expression successively and putting $x = a$, it is easy to see that

$$F(a) = 0, \quad F'(a) = 0, \quad F^2(a) = 0 \;\ldots\ldots\; F^n(a) = 0.$$

Hence, by Lemma XX, Cor. 2, 0 must be the limiting value of $\dfrac{F(x)}{(x-a)^n}$ when x approaches a, and therefore $\dfrac{F(x)}{(x-a)^n}$ may be diminished *ad libitum* by sufficiently diminishing $x \sim a$. If therefore we put

$$\frac{F(x)}{(x-a)^n} = \frac{Q}{\Gamma n}, \text{ or } F(x) = Q \cdot \frac{(x-a)^n}{\Gamma n},$$

and substitute for $F(x)$ its value (1) we have

$$f(x) - \left\{f(a) + f'(a)\frac{x-a}{\Gamma 1} + f^2(a)\frac{(x-a)^2}{\Gamma 2} \ldots\ldots\right.$$

$$\left. f^r(a)\frac{(x-a)^n}{\Gamma n}\right\} = Q\,\frac{(x-a)^n}{\Gamma n},$$

$$\text{or } f(x) = f(a) + f'(a)\frac{x-a}{\Gamma 1} + f^2(a)\frac{(x-a)^2}{\Gamma 2}\ldots\ldots$$

$$+\left\{f^n(a) + Q\right\}\frac{(x-a)^n}{\Gamma n},$$

where Q, since it $= \Gamma n\,\frac{F(x)}{(x-a)^n}$, is some function of x which may be diminished *ad libitum* by sufficiently diminishing $x \sim a$. Q. E. D.

This reasoning fails when any of the quantities $f(a)$, $f'(a) \ldots f^n(a)$ are infinite; for then we cannot assert that all the quantities $F(a)$, $F'(a)$, $F''(a) \ldots F^n(a)$ are zero; which is essential to the proof.

Lemma XXII. †† 123. *If* M *be the least value of* $\mathrm{f}^n(\mathrm{x}) - \mathrm{f}^n(\mathrm{a})$, *and* N *the greatest, for all values of* x *between* a *and another value* b*; then* Q *lies between* M *and* N, *for all values of* x *between* a *and* b.

Let C be any constant, and let us write down the expression $F(x) - C\,\frac{(x-a)^n}{\Gamma n}$ and its successive differential coefficients as follows:

$$F(x) - C\frac{(x-a)^n}{\Gamma n} \ldots\ldots (n),$$

$$F'(x) = C\frac{(x-a)^{n-1}}{\Gamma(n-1)} \ldots\ldots (n-1),$$

$$F^2(x) - C\frac{(x-a)^{n-2}}{\Gamma(n-2)} \ldots\ldots (n-2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$F^{n-1}(x) - C(x-a) \ldots\ldots\ldots\ldots (1),$$

$$F^n(x) - C \ldots\ldots\ldots\ldots\ldots\ldots\ldots (0).$$

Now $F^n(x) = f^n(x) - f^n(a)$ evidently; hence putting $C = M$, which is the least value of $f^n(x) - f^n(a)$ and therefore of $F^n(x)$ for all values of x between a and b, it is clear that if we suppose x to increase from a to b, (supposing b greater than a,) the expression (0) is always positive, and therefore the expression (1) is always increasing by Lemma XVIII, but (1) is zero when $x = a$, therefore (1) is always positive; and therefore in the same way we may shew that (2) is always positive; and so on, and finally that (n) is always positive, and therefore that $\Gamma n \dfrac{F^n(x)}{(x-a)^n}$, or Q, is always greater than C, i.e. M.

In exactly the same way if we put $C = N$, we may shew that the expressions (0), (1), (2) ... (n) are all negative while x increases from a to b, and that therefore Q is always less than N.

If b be less than a we may shew in exactly the same way that when x decreases from a to b the expressions (0), (1), (2) ..., and finally (n) are all negative if $C = M$, and positive when $C = N$; and therefore that Q always lies between M and N.

Hence it appears that Q lies between M and N for all values of x between a and b. Q. E. D.

Lemma XXIII. 124. *When a function* f(x) *becomes infinite for a particular value of* x, *all its differential coefficients must also become infinite.*

For $\dfrac{f(x) - f(a)}{x - a}$ may be made to differ as little as we please from $f'(a)$ by making x approach a; hence if $f(a)$ and therefore $\dfrac{f(x) - f(a)}{x - a}$ be infinite, and at the same time $f'(a)$ finite, we may make infinity differ as little as we please from a finite quantity; which is absurd. Hence $f'(a) = \infty$: and therefore by similar reasoning $f^2(a) = \infty$ and $f^3(a) = \infty$, and so on.

This reasoning does not hold if $a = \infty$.

CHAPTER IX.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO THE DEVELOPEMENT OF FUNCTIONS; WITH SOME PRELIMINARY REMARKS RESPECTING SERIES.

We now proceed to shew some of the principal uses which may be made of the differential calculus in pure mathematics, and in the first place its application to the developement of functions in series.

Preliminary remarks respecting series.

125. It will be necessary to make a few preliminary remarks on the nature of an ordinary series of the form

$$A_0 + A_1 x + A_2 x^2 \ldots + A_n x^n + \&c. \ldots ;$$

and to settle precisely what we mean by the equation

$$f(x) = A_0 + A_1 x + A_2 x^2 \ldots + A_n x^n + \&c. \ldots (1)$$

which expresses the developement of a function $f(x)$ in a series of powers of x.

= always denotes actual equality.

126. In the first place we may remark that the sign = here, as well as elsewhere, always signifies actual equality; for mathematicians always consider themselves at liberty, when they have two expressions of any kind connected by the sign =, to use them indifferently for each other, and to substitute one for the other in any calculation; which certainly they have no right to do, if the sign = does not always signify actual equality. We shall therefore always use = as the sign of actual equality.

We must not *in general* suppos a series to consist of an infinite number of

127. Secondly, by the second member of the equation (1) we do not *in general* mean a series of terms infinite in number, but a series of terms carried on according to a certain law to any number we please, with a term at the end of a dif-

terms, but of a finite number with a remainder.

ferent kind from the rest, which is commonly called a remainder: the former terms are represented by

$$A_0 + A_1 x + A_2 x \ldots + A_n x^n,$$

n being any integer, and the remainder is represented by the "+ &c.," which is written after the term $A_n x^n$. The remainder we shall often denote by writing $+ R$ instead of "+ &c."

Thus when we say that

$$\frac{1}{1-x} = 1 + x + x^2 \ldots + x^n + \&\text{c.} \ldots\ldots (2)$$

we do not *in general* mean to assert that $1 + x + x^2 \ldots$ to an infinite number of terms $= \dfrac{1}{1-x}$, for if so, suppose $x = 2$ and then we have $1 + 2 + 8 + 16 \ldots$ to an infinite number of terms $= -1$ which is manifestly absurd. But we simply mean to assert that

$$\frac{1}{1-x} = 1 + x + x^2 \ldots + x^n + R \ldots\ldots (3)$$

where n is a number as large as we please, and R a certain quantity which must be always added to the series to make the second member of the equation equal to the first. In fact the "+ &c." in equation (2), and the $+ R$ in equation (3) mean the same thing.

In the present instance the value of R is $\dfrac{x^{n+1}}{1-x}$, for it is easy to see that

$$\frac{1}{1-x} = 1 + x + x^2 \ldots x^n + \frac{x^{n+1}}{1-x},$$

and thus when $x = 2$ we have

$$-1 = 1 + 2 + 8 + 16 \ldots 2^n - 2^{n+1},$$

in which equation there is no such absurdity as there is when the number of terms is considered infinite.

We may under cer-

128. Thirdly, if it should so happen that R is a quantity

which may be diminished *ad libitum* by sufficiently increasing n, then we may assert that

tain circumstances assert that a series consists of an infinite number of terms.

$$f(x) = A_0 + A_1 x + A_2 x^2 \dots$$

to an infinite number of terms: or perhaps more correctly, that $f(x)$ is the limiting value of $A_0 + A_1 x + A_2 x^2 \dots A_n x^n$ when n approaches infinity.

For $f(x) - (A_0 + A_1 x + A_2 x^2 \dots A_n x^n)$, since it $= R$, may be diminished *ad libitum* by sufficiently increasing n; and therefore, by Lemma VI, Cor. 2, $f(x)$ is the limiting value of $A_0 + A_1 x + A_2 x^2 \dots A_n x^n$ when n approaches ∞. When we assert therefore that

$$f(x) = A_0 + A_1 x + A_2 x^2 \dots \textit{ad infinitum},$$

we simply mean that $f(x)$ is the limiting value of

$$A_0 + A_1 x + A_2 x^2 + \dots A_n x^n,$$

when n approaches ∞.

Definition of a converging and of a diverging series.

129. When R may be diminished *ad libitum* by sufficiently increasing n, the series

$$A_0 + A_1 x + A_2 x^2 \dots A_n x^n + R$$

is said to be a *converging series:* otherwise it is called a *diverging series.*

Hence we may assert that

$$f(x) = A_0 + A_1 x + A_2 x^2 \dots \textit{ad infinitum},$$

when the series is converging, but not otherwise.

A function may be developed in an infinite number o different series.

130. In the fourth place we may remark, that we may always assume

$$f(x) = A_0 + A_1 x + A_2 x^2 \dots A_n x^n + R,$$

where A_0, A_1, $A_2 \dots A_n$ are any constants whatever. For we may always give R such a value as will make the second member of this equation to coincide with the first, no matter what values we assign to A_0, A_1, $A_2 \dots A_n$.

Thus we may assume

$$\frac{1}{1-x} = 1 + x + x^2 + 5x^3 - 7x^4 + R,$$

for by giving R the value $-\frac{4x^3 - 12x^4 + 7x^5}{1-x}$ the second member of this equation becomes identical with the first.

We may therefore expand any function $f(x)$ in an infinite number of different ways in a series of the form

$$A_0 + A_1x + A_2x^2 \ldots A_nx^n + R \ldots (1).$$

Distinction between a perfect and an imperfect developement.

131. And here, in the fifth place, we may make an important distinction; namely, the distinction between what we may call a *perfect* and an *imperfect developement.*

The series $A_0 + A_1x + A_2x^2 \ldots A_nx^n + R$ we shall call a perfect developement of $f(x)$, when R is of such a nature, that zero is the limiting value of $\frac{R}{x^n}$ when x approaches zero; and if this limiting value be not zero we shall call the series an imperfect developement.

That we are justified in making this distinction, and regarding it as a very important distinction, is evident from the following theorem.

The principle of indeterminate coefficients holds for a perfect but not for an imperfect developement.

132. *The principle of indeterminate coefficients holds for a perfect developement, but not for an imperfect.*

By the principle of indeterminate coefficients we mean this: that if we have a series

$$A_0 + A_1x + A_2x^2 \ldots + A_nx^n + R,$$

which is proved to be zero for all values of x, then must

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = 0 \ \ldots \ A_n = 0.$$

Let $A_0 + A_1x + A_2x^2 \ldots A_nx^n + R$ be a perfect series, which is proved to be zero for all values of x, then must

$$\frac{R}{x^n} = -\frac{A_0 + A_1x + A_2x^2 \ldots A_nx^n}{x^n},$$

for all values of x, except of course the value zero: but the series being perfect, the limiting value of $\frac{R}{x^n}$, when x approaches 0 is zero; hence, by Lemma VI, Cor. 1, zero must be the limiting value of

$$\frac{A_0 + A_1 x + A_2 x^2 \ldots A_n x^n}{x^n}$$

when x approaches 0: now if A_0 be not zero,

$$\frac{A_0 + A_1 x \ldots A_n x^n}{x^n} \text{ becomes } \frac{A_0}{0} \text{ or } \infty,$$

when $x = 0$, and its limiting value cannot be zero when x approaches zero; therefore $A_0 = 0$, and we have

$$\frac{A_0 + A_1 x \ldots A_n x^n}{x^n} = \frac{A_1 + A_2 x \ldots A_n x^{n-1}}{x^{n-1}},$$

except of course when x actually $= 0$.

Hence zero must be the limiting value of this latter quantity when x approaches 0, which as before cannot be true unless $A_1 = 0$: and so we may go on and prove that $A_3 = 0$, $A_4 = 0 \ldots$; and lastly, that

$$\frac{A_0 + A_1 x \ldots A_n x^n}{x^n} = A_n,$$

the limiting value of which cannot be zero unless $A_n = 0$. Hence the principle of indeterminate coefficients holds for a perfect series.

But the principle of indeterminate coefficients does not hold for an imperfect developement.

For by what we have just proved, it is evident that if A_0, A_1, $A_2 \ldots A_n$ be each zero, then the limiting value of $\frac{R}{x^n}$ when x approaches zero must be zero, and therefore the series must be perfect. If therefore the series be not perfect, A_0, $A_1 \ldots A_n$ cannot be each zero, i. e. the principle of indeterminate coefficients does not hold.

Hence I think the importance of distinguishing series into perfect and imperfect is manifest.

A function which has only one value for each value of x can be developed in only perfect series.

133. Lastly, we may remark, that if $f(x)$ be a function which has only one value for each value of x, it cannot be developed in more than one perfect series.

For if possible let the two different perfect series

$$A_0 + A_1 x + A_2 x^2 \ldots A_n x^n + R, \text{ and } B_0 + B_1 x + B_2 x^2 \ldots B_n x^n + S,$$

be both equal to $f(x)$, then since $f(x)$ has only one value for each value of x, these series must be equal to each other; and therefore we have

$$A_0 - B_0 + (A_1 - B_1)\,x + (A_2 - B_2)\,x^2 \ldots + (A_n - B_n)\,x^n + R - S = 0.$$

Now when x approaches 0, the limiting values of $\frac{R}{x^n}$ and of $\frac{S}{x^n}$ are zero, therefore so also is that of $\frac{R-S}{x^n}$: hence, by what has been proved in the preceding Article, we must have $A_0 = B_0$, $A_1 = B_1$, $A_2 = B_2 \ldots A_n = B_n$, and therefore $R = S$; hence the two series are identical; and therefore $f(x)$ cannot be developed in two different perfect series.

Having made these preliminary remarks, we now proceed to shew the manner in which the Differential Calculus may be applied to the developement of functions in perfect series.

Proposition. To develope $f(a+h)$ in a perfect series of powers of h.

134. *To develope* f (a + h) *in a perfect series of powers of* h.

In Lemma XXI. put $a + h$ for x, and we have

$$f(a+h) = f(a) - f'(a)\frac{h}{\Gamma 1} + f^2(a)\frac{h^2}{\Gamma 2} \ldots f^n(a)\frac{h^n}{\Gamma n} + Q\frac{h^n}{\Gamma n} \quad (1),$$

here $Q\frac{h^n}{\Gamma n}$ is the remainder; call it R, then $\frac{R}{h^n} = \frac{Q}{\Gamma n}$; now zero is the limiting value of Q when x approaches a, i. e. when h approaches zero, (see Lemma XXI.), and therefore the same is true of $\frac{R}{h^n}$; hence (1) is a perfect developement. Since

Lemma XXI. fails when any of the quantities $f(a)$, $f'(a)$, $f^2(a) \ldots f^n(a)$ become infinite, this developement fails under the same circumstances.

This developement is known by the name of Taylor's Series; when $a = 0$ it is called Maclaurin's Series, in which case it becomes

Taylor's and Maclaurin's series.

$$f(h) = f(0) + f'(0)\frac{h}{\Gamma 1} + f^2(0)\frac{h^2}{\Gamma 2} \ldots + f^n(0)\frac{h^n}{\Gamma n} + Q\frac{h^n}{\Gamma n}.$$

We may write the developement (1) thus:

$$f(a+h) = f(a) + f'(a)\frac{h}{\Gamma 1} + f^2(a)\frac{h^2}{\Gamma 2} \ldots + f^n(a)\frac{h^n}{\Gamma n} + \&c.$$

remembering what "+ &c." means.

The following are very important examples of this developement.

Examples.

135. Let $f(x) = x^m$; and therefore

Developement of $(1+x)^m$.

$$f^n(x) = m(m-1)(\ldots)(m-n+1)x^{m-n};$$

then putting 1 for a in the developement (1), we find

$$(1+h)^m = 1 + \frac{m}{1}h + \frac{m(m-1)}{\Gamma 2}h^2 \ldots$$
$$+ \frac{m(m-1)(\ldots.)(m-n+1)}{\Gamma n}h^n + \&c.$$

which is the binomial theorem for any value of the index.

Let $f(x) = \log x$, and $\therefore f^n(x) = \Gamma(n-1)(-x)^{-n}$; then putting 1 for a in the developement (1), we find

Developement of $\log(1+x)$.

$$\log(1+h) = \frac{h}{1} - \frac{h^2}{2} + \frac{h^3}{3} \ldots + \frac{(-h)^n}{n} + \&c.$$

Let $f(x) = c^x$, and $\therefore f^n(x) = k^n c^x$, where $k = \log c$, then putting $a = 0$ in the developement (1), we find

Developement of c^x.

$$c^h = 1 + \frac{kh}{\Gamma 1} + \frac{k^2h^2}{\Gamma 2} \ldots + \frac{k^n h^n}{\Gamma n} + \&c.$$

Developement of $\sin x$, $\cos x$, $\tan^{-1} x$.

Let $f(x) = \sin x$, and $\therefore\ f^n(x) = \sin\left(x + \frac{n\pi}{2}\right)$; then putting $a = 0$ in the developement (1), we find

$$\sin h = \frac{h}{\Gamma 1} - \frac{h^3}{\Gamma 3} + \frac{h^5}{\Gamma 5} \ldots + (-)^n \frac{h^{2n+1}}{\Gamma(2n+1)} + \&c.$$

Similarly we find

$$\cos h = 1 - \frac{h^2}{\Gamma 2} + \frac{h^4}{\Gamma 4} \ldots + (-)^n \frac{h^{2n}}{\Gamma(2n)} + \&c.$$

$$\tan^{-1} h = \frac{h}{1} - \frac{h^3}{3} + \frac{h^5}{5} \ldots + (-)^n \frac{h^{2n+1}}{\Gamma(2n+1)} + \&c.$$

For more examples of the use of this developement, see Appendix G.

Proposition. Estimation of the error committed when the remainder in Taylor's series is neglected.

††136. *If we neglect the remainder* $\mathrm{Q}\frac{\mathrm{h}^{\mathrm{n}}}{\Gamma \mathrm{n}}$ *in the developement* (1) *last article, the error we commit in so doing lies between* $\mathrm{M}\frac{\mathrm{h}^{\mathrm{n}}}{\Gamma \mathrm{n}}$ *and* $\mathrm{N}\frac{\mathrm{h}^{\mathrm{n}}}{\Gamma \mathrm{n}}$, *where* M *and* N *are the greatest and least values of* $\mathrm{f}^{\mathrm{n}}(\mathrm{x}) - \mathrm{f}^{\mathrm{n}}(\mathrm{a})$ *for all values of* x *between* a *and* a + h.

This follows immediately from Lemma XXII, if we put $b = a + h$: for it is there proved that Q lies between M and N for all values of x between a and b (i. e. $a + h$), and therefore for the particular value $a + h$; consequently the remainder $Q\frac{h^n}{\Gamma n}$ in the developement (1) must lie between $M\frac{h^n}{\Gamma n}$ and $N\frac{h^n}{\Gamma n}$.

Examples.

Ex. 1. Let $f(x) = \log x$, $a = 1$, $h = \frac{1}{10}$, $n = 5$; then $f^n(x) = (-)^{n-1}\frac{\Gamma(n-1)}{x^n}$; $\therefore\ f^5(x) - f^5(1) = \Gamma 4 \cdot \left(\frac{1}{x^5} - 1\right)$: the greatest and least values of this for all values of x between

1 and $1+\frac{1}{10}$ are evidently 0 and $\Gamma 4 \left\{\frac{1}{\left(1+\frac{1}{10}\right)^5}-1\right\}$. Hence the error lies between 0 and $\frac{1}{5}\left(\frac{1}{11^5}-\frac{1}{10^5}\right)$; and we have therefore

$$\log\left(1+\frac{1}{10}\right)=\frac{1}{10}-\frac{1}{2}\frac{1}{10^2}+\frac{1}{3}\frac{1}{10^3}-\frac{1}{4}\frac{1}{10^4}+\frac{1}{5}\frac{1}{10^5}$$

$$+ \text{ an error between 0 and } \frac{1}{5}\left(\frac{1}{11^5}-\frac{1}{10^5}\right).$$

Since $\frac{1}{11^5}=.000006\ldots$ this error lies between 0 and $-\frac{.000004\ldots}{5}$ which has no significant digit in the first 6 decimal places.

Ex. 2. Let $f(x)=x^{\frac{1}{2}}$, $a=1$, $h=\frac{1}{10}$, $n=5$; then

$$f^n(x)=\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)(\ldots)\left(\tfrac{1}{2}-n+1\right)x^{\frac{1}{2}-n};$$

$$\therefore f^5(x)-f^5(1)=\frac{1.3.5.7}{2^5}\left(\frac{1}{x^{\frac{9}{2}}}-1\right):$$

the greatest and least values of this for all values of x between 1 and $1+\frac{1}{10}$ are 0 and

$$\frac{1.3.5.7}{2^5}\left\{\frac{1}{\left(1+\frac{1}{10}\right)^{\frac{9}{2}}}-1\right\}.$$

Hence the error lies between 0 and

$$\frac{1.3.5.7}{2^5\,\Gamma 5}\left\{\frac{1}{\left(1+\frac{1}{10}\right)^{\frac{9}{2}}}-1\right\}\frac{1}{10^5};$$

$$\text{or } -\frac{7}{156}\left\{\frac{1}{10^5}-\frac{1}{11^5}\cdot\left(\frac{11}{10}\right)^{\frac{1}{2}}\right\},$$

which is evidently, in *numerical* value, less than $\frac{1}{20}\left(\frac{1}{10^5}-\frac{1}{11^5}\right)$ or $\frac{.000004\ldots}{20}$ or $.0000002\ldots$. Hence we have

$$\sqrt{1+\frac{1}{10}}=1+\frac{1}{2}\frac{1}{10}-\frac{1}{2^3}\frac{1}{10^2}+\frac{1}{2^4}\frac{1}{10^3}-\frac{5}{2^7}\frac{1}{10^4}+\frac{7}{2^8}\frac{1}{10^5}$$

$$+ \text{ an error between } 0 \text{ and } -.0000002\ldots.$$

Ex. 3. Let $f(x)=e^x$, $a=0$, $h=1$, $n=10$; then

$$f^n(x)=e^x; \quad \therefore f^{10}(x)-f(0)=e^x-1:$$

the greatest and least values of this for all values of x between 0 and 1 are $e-1$ and 0. Hence the error lies between $\frac{e-1}{\Gamma(10)}$ and 0, or between $\frac{2}{\Gamma(10)}$ and 0 since e is <3; therefore

$$e=1+\frac{1}{\Gamma 1}+\frac{1}{\Gamma 2}+\frac{1}{\Gamma 3}\cdots\frac{1}{\Gamma(10)}$$

+ an error between 0 and $\frac{2}{\Gamma(10)}$, which has no significant digit in the first 6 decimal places.

Condition of convergency.

†† 137. Let u_n denote the *numerical* value of

$$f^n(x)\frac{(x-a)^n}{\Gamma n},$$

for any value of x between a and $a+h$; then if the limiting value of u_n when n approaches ∞ is zero, the developement (1) is converging, and we may assert that

$$f(a+h)=f(a)+f^1(a)\frac{h}{\Gamma 1}+f^2(a)\frac{h^2}{\Gamma 2}$$

$$+f^3(a)\frac{h^3}{\Gamma 3}\ldots \textit{ad infinitum}.$$

For then it is clear that

$$\{f^n(x) - f^n(a)\} \frac{(x-a)^n}{\Gamma n},$$

and therefore $M\dfrac{h^n}{\Gamma n}$, $N\dfrac{h^n}{\Gamma n}$, and therefore $Q\dfrac{h^n}{\Gamma n}$ may be diminished *ad libitum* by sufficiently increasing n: therefore by (130) the developement is converging, and we may assert that

$$f(a+h) = f(a) + f^1(a)\frac{h}{\Gamma 1} + f^2(a)\frac{h^2}{\Gamma 2}$$

$$+ f^3(a)\frac{h^3}{\Gamma 3} \dots \text{ ad infinitum.}$$

A simpler condition of convergency.

††138. If, for all values of n greater than a certain value r, $\dfrac{u_{n+1}}{u_n}$ never exceeds a certain ratio a which is less than unity, the developement is converging.

For then we have $\dfrac{u_{r+1}}{u_r}$ not $> a$, $\dfrac{u_{r+2}}{u_{r+1}}$ not $> a$, $\dfrac{u_{r+3}}{u_{r+2}}$ not $> a \dots\dots \dfrac{u_n}{u_{n-1}}$ not $> a$; and therefore, multiplying these inequalities, we have $\dfrac{u_n}{u_r}$ not $> a^{n-r}$, or u_n not $> a^{n-r}u_r$. Now since a is less than unity, the limiting value of a^{n-r} when n approaches ∞ is zero, and therefore, zero is also the limiting value of $a^{n-r}u_r$, and therefore of u_n: consequently the developement is converging.

Taylor's series arrived at somewhat differently.

139. By means of Lemma XXI. we arrive at the conclusion, that there necessarily exists a perfect developement of $f(a+h)$ in the form

$$A_0 + A_1h + A_2h^2 \dots + A_nh^n + R,$$

provided none of the quantities $f(a)$, $f^1(a)$, $f^2(a) \dots f^n(a)$ be infinite: we shall now prove the same thing somewhat differently.

By (131) we may assume that

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 \ldots + A_n(x-a)^n + R \ldots . (1),$$

A_0, $A_1 \ldots A_n$ being any arbitrary constants, and R that quantity, whatever it be, which must be added to the second member of the equation to make it equal to the first. If we can so determine $A_0, A_1, A_2 \ldots A_n$ that the limiting value of $\dfrac{R}{(x-a)^n}$ when x approaches a shall be zero, then the developement (1) will be a perfect developement. Now if this limiting value $= 0$, it follows from Lemma XX. Cor. 3, that

$$\frac{dR}{dx}, \quad \frac{d^2R}{dx^2}, \quad \frac{d^3R}{dx^3} \ldots . \frac{d^nR}{dx^n}$$

must be each zero when $x = a$. Hence, differentiating (1) put in the form

$$f(x) - A_0 - A_1(x-a) - A_2(x-a) \ldots . - A_n(x-a) = R$$

n times successively, and then putting $x = a$, we obtain the following equations:

$$\left.\begin{array}{l} f(a) - \quad A_0 = 0 \\ f'(a) - \Gamma 1 . A_1 = 0 \\ f^2(a) - \Gamma 2 . A_2 = 0 \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ f^n(a) - \Gamma n . A_n = 0 \end{array}\right\} \ldots\ldots\ldots (2),$$

which equations give us the values of A_0, A_1, $A_2 \ldots A_n$, necessary to make (1) a perfect developement.

And conversely, if we give the values (2) to A_0, A_1, $A_2 \ldots A_n$, we shall evidently have

$$R = 0, \quad \frac{dR}{dx} = 0, \quad \frac{d^2R}{dx^2} = 0 \ldots \frac{d^nR}{dx^n} = 0, \text{ when } x = a;$$

and therefore the limiting value of $\dfrac{R}{(x-a)^n}$ when x approaches a, will (by Lemma XX. Cor. 2.) be zero; i.e. the developement will be perfect.

Hence it appears (putting $x - a = h$) that

$$f(a) + f'(a)\frac{h}{\Gamma 1} + f^2(a)\frac{h^2}{1.2} \ldots\ldots f^n(a)\frac{h^n}{\Gamma n} + R$$

is a perfect developement, and the only perfect developement, of $f(a+h)$.

We must except however the case where any of the quantities $f(a)$, $f'(a)$, $f^2(a) \ldots f^n(a)$ are infinite, for then we cannot be sure that the equations (2) are satisfied by giving A_0, A_1, $A_2 \ldots$ &c. the above values; for instance, if $f^3(a) = \frac{1}{0}$, we cannot assert that $f^3(a) - \Gamma 3 \,.\, A_3 = 0$, if we give A_3 the value $\frac{1}{0}$, for then the quantity $f^3(a) - \Gamma 3 \,.\, A_3$ assumes the form $\frac{1}{0} - \frac{1}{0}$, which we cannot assert is zero.

140. If $f(x)$, and all its differential coefficients below a certain one, the p^{th} suppose, be finite, but the p^{th} (and all above it by Lemma XXIII.) infinite; then the developement (1) holds as a perfect developement for all values of n less than p, but for all other values of n it fails.

The failure of Taylor's series.

141. When a failure of this kind takes place, it is generally possible to obtain a perfect developement containing fractional powers of h after the p^{th} term.

A differential coefficient becoming infinite when $x = a$, indicates the appearance of fractional powers in the developement.

For let
$$F(x) = f(x) - f(a) - f'(a)\frac{x-a}{\Gamma 1}$$
$$- f^2(a)\frac{(x-a)^2}{\Gamma 2} \ldots - f^{p-1}(a)\frac{(x-a)^{p-1}}{\Gamma(p-1)}, \text{ see } (123)$$

then it is clear that

$$F(a) = 0, \quad F'(a) = 0, \quad F^2(a) = 0 \ldots\ldots F^{p-1}(a) = 0,$$

but $F^p(a) = f^p(a) = \infty$; since we suppose that $f(a)$, $f'(a)$, $f^2(a) \ldots\ldots f^{p-1}(a)$ are finite quantities, but $f^p(a)$, $f^{p+1}(a) \ldots$ infinite.

Hence, by Lemma XX. Cor. 2, the limiting value of $\frac{F(x)}{(x-a)^{p-1}}$ is zero, and that of $\frac{F(x)}{(x-a)^p}$ is infinite, when x approaches a.

Now if any function $\phi(x) = 0$ when $x = a$, we may generally find some power of $x - a$, $(x-a)^m$ suppose, such that the limiting value of $\frac{\phi(x)}{(x-a)^m}$ when x approaches a, is neither zero nor infinity. Let us then suppose $(x-a)^m$ to be such a power of $x - a$, that the limiting value of $\frac{F(x)}{(x-a)^m}$ when x approaches a is some constant C, which is neither zero nor infinity: then, since

$$\frac{F(x)}{(x-a)^{p-1}} = \frac{F(x)}{(x-a)^m}.\,(x-a)^{m-p+1}$$

$$\text{and } \frac{F(x)}{(x-a)^p} = \frac{F(x)}{(x-a)^m}\,(x-a)^{m-p},$$

it is clear that the limiting values of $\frac{F(x)}{(x-a)^{p-1}}$ and $\frac{F(x)}{(x-a)^p}$ are $C.0^{m-p+1}$ and $C.0^{m-p}$. If m be less than $p-1$ the former of these is infinite, which we know not to be the case; and if m be greater than p the latter will be zero, which we know not to be the case: hence m lies between $p - 1$ and p, and is therefore a fraction.

If therefore we assume $\frac{F(x)}{(x-a)^m} = C + Q$, Q will be some quantity which becomes zero when $x = 0$, and we have, putting for $F(x)$ its value

$$f(x) = f(a) + f'(a)\frac{(x-a)}{\Gamma 1} + f^2(a)\frac{(x-a)^2}{\Gamma 1} \ldots\ldots$$

$$f^{p-1}(a)\frac{(x-a)^{p-1}}{\Gamma(p-1)} + (C+Q)(x-a)^m,$$

which is a perfect developement containing a fractional power $C(x-a)^m$, m being greater than $p - 1$, and less than p.

Thus it appears that when any differential coefficient $f^p(a)$ becomes infinite, it indicates the appearance of a fractional power in the developement of $f(x)$ in powers of $x - a$.

How the developement is to be obtained when any of the differential coefficients become infinite.

142. We may generally apply Taylor's series, in the following manner, to determine the developement of $f(x)$, when fractional powers appear in it in consequence of some of its differential coefficients becoming infinite when $x = a$.

Put $x = a + z^r$, and $f(x)$ will become a function of z, $\phi(z)$ suppose: then, if possible, so determine r that neither $\phi(z)$ nor any of its differential coefficients shall become infinite when $z = 0$, and this being the case we shall have, by Taylor's Theorem,

$$\phi(z) = \phi(0) + \phi'(0)\frac{z}{\Gamma 1} + \phi^2(0)\frac{z^2}{\Gamma 2} \dots \phi^n(0)\frac{z^n}{\Gamma n} + R,$$

n being as large as we please.

Now here put for z its value $(x - a)^{\frac{1}{r}}$, and for $\phi(z)$ its value $f(x)$, and we have

$$f(x) = \phi(0) + \phi'(0)\frac{(x-a)^{\frac{1}{r}}}{\Gamma 1}$$
$$+ \phi^2(0)\frac{(x-a)^{\frac{2}{r}}}{\Gamma 2} \dots \phi^n(0)\frac{(x-a)^{\frac{n}{r}}}{\Gamma n} + R,$$

which is the developement required containing fractional powers.

There are also other ways of substituting for x, so as to obtain the developement of $f(x)$ by Taylor's series.

Ex. Let $f(x) = \sin\left\{x + 1 + (x-1)^{\frac{3}{2}}\right\}$; Example.

here it is easy to see that $f^2(0) = \infty$; therefore we shall not be able to develope $f(x)$ in this case in integral powers of $x - 1$. To obtain the developement by means of Taylor's Theorem, put $x = 1 + z^2$, and then

$$f(x) = \sin(2 + z^2 + z^3) = \phi(z),$$

and it is easy to see that neither $\phi(z)$ nor any of its differential coefficients become infinite when $z = 0$: therefore we have

$$f(x) = \phi(0) + \phi'(0)\frac{(x-1)^{\frac{1}{2}}}{\Gamma 1} + \phi^2(0)\frac{(x-1)}{\Gamma 2} + \phi^3(0)\frac{(x-1)^{\frac{3}{2}}}{\Gamma 3} + \&c.$$

By actual differentiation, and putting $z = 0$, we find

$$\phi(0) = \sin 2,\ \ \phi^1(0) = 0,\ \ \phi^2(0) = 2\cos 2,\ \ \phi^3(0) = 6\cos 2 \ldots \&c.$$

and therefore,

$$\sin\left\{x + 1 + (x-1)^{\frac{3}{2}}\right\} = \sin 2 + \cos 2 . (x-1) + \cos 2 . (x-1)^{\frac{3}{2}} + \&c.$$

Same example done more simply.

This method however is not generally the simplest in practice; other substitutions for x often bring out the result more readily. Thus assume

$$x - 1 + (x-1)^{\frac{3}{2}} = z,$$

and then $f(x) = \sin(2 + z) = \phi(z)$, suppose;

and it is evident that neither $\phi(z)$ nor any of its differential coefficients become infinite when $x = 1$; i. e. when $z = 0$; we have therefore

$$\phi(z) = \phi(0) + \phi'(0)\frac{z}{\Gamma 1} + \phi^2(0)\frac{z^2}{\Gamma 2} + \&c.$$

Now since $\phi(z) = \sin(2 + z)$, we have

$$\phi(0) = \sin 2,\quad \phi'(0) = \cos 2,\quad \phi^2(0) = -\sin 2,$$
$$\phi^3(0) = -\cos 2 \ldots \&c.;$$

and therefore, restoring for z its value, we have

$$f(x) = \sin 2 + \cos 2 . \left\{(x-1) + (x-1)^{\frac{3}{2}}\right\}$$
$$- \sin 2 . \left\{(x-1) + (x-1)^{\frac{3}{2}}\right\}^2 \ldots\ldots$$

if we arrange this in powers of $(x - 1)$, expanding each term by the binomial theorem, we find the developement required.

$f(a)$ being infinite indicates the

143. These methods clearly suppose that $f(a)$ is not infinite.

appearance of negative powers in the series.

If $f(a) = \infty$, then if we assume

$$(x-a)^s f(x) = \phi(x),$$

and so determine s that $\phi(a)$ shall not be infinite, we may develope $\phi(x)$ as above in the form

$$\phi(x) = A_0 + A_1 (x-a)^{\frac{1}{r}} + A_2 (x-a)^{\frac{2}{r}} \ldots \&c.$$

and therefore, restoring $f(x)$, we have

$$f(x) = A_0 (x-a)^{-s} + A_1 (x-a)^{-s+\frac{1}{r}} \ldots \&c.$$

Hence, $f(x)$ becoming infinite when $x = a$, indicates the appearance of negative powers in the series. (For examples, see Appendix H.)

Developement of a function given by an equation.

†† 144. *Suppose we have a relation between* x *and* y, *viz.* f (x y) = 0, *to develope* y *in a perfect series of powers of* x.

To do this all that is necessary in general is, to write down to the equation $f(xy)$ and its successive derivatives, and then to put $x = 0$, and so find the values of $y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$, &c. when $x = 0$, and thus obtain the perfect developement of y in powers of x by Taylor's Theorem. An example will best explain this method.

Example.

Let the relation between x and y be

$$y^3 - xy + 2x = 1 \ldots (1);$$

then differentiating successively, we have

$$(3y^2 - x)\,p - y + 2 = 0 \ldots (2),$$

$$(3y^2 - x)\,q + 6yp^2 - 2p = 0 \ldots (3),$$

when $p, q, r \ldots$ are, for brevity, put for $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3} \ldots$

$$(3y^2 - x)\,r + (18yp - 3)\,q + 6p^3 = 0 \ldots (4)$$

....

&c. &c.

Hence putting $x = 0$, we find

from (1) $y^3 = 1$; $\therefore y = 1$, (at least 1 is one of the 3 values of y);

$\therefore$ from (2) $3p + 1 = 0$; $\therefore p = -\frac{1}{3}$;

$\therefore$ from (3) $3q + \frac{6}{9} + \frac{2}{3} = 0$; $\therefore q = -\frac{4}{9}$;

$\therefore$ from (4) $3r + 9 \cdot \frac{4}{9} + \frac{2}{9} = 0$; $\therefore r = -\frac{38}{27}$;

……………………………………

&c. &c.

Hence by Taylor's series, we have

$$y = 1 - \frac{1}{3}\frac{x}{\Gamma 1} - \frac{4}{9}\cdot\frac{x^2}{\Gamma 2} - \frac{38}{27}\frac{x^3}{\Gamma 3} \dots \&c.$$

How we must proceed when Taylor's series fails.

†† 145. It often happens however that we thus get infinite values for some of the quantities p, q, r, &c., and of course this method fails; we must then proceed as follows.

Let a be a value (or one of the values) of y when $x = 0$, then substitute $a + ux^m$ for y in the given equation $f(xy) = 0$, and, if possible, give m such a value, that the limiting value of u when x approaches 0 shall not be 0 or ∞, but some quantity, b suppose. Then assume $u = b + vx^n$, substitute this value of u in the equation and so determine n, if possible, that the limiting value of v when x approaches zero shall not be 0 or ∞, but some quantity, c suppose. Then assume $v = c + wx^p$; it is clear that in this manner we obtain a perfect developement for y, viz.,

$$y = a + bx^m + cx^{m+n} + R, \quad \text{putting } R = wx^{m+n+p},$$

which we may carry as far as we please.

Several different series

It generally happens that we obtain more than one value of y when $x = 0$, and more than one value of m, or of n, or of r, &c., which satisfies the above conditions, in which case we find more than one perfect developement for y. generally obtained in this manner.

An example will be necessary to make the nature of this process of developement clear. Example.

Let $y^3 - 3axy + x^3 = 0$,

then $y = 0$ when $x = 0$; assume $\therefore$ $y = ux^m$,

and $\therefore$ substituting

$$u^3 x^{3m} - 3aux^{m+1} + x^3 = 0 \ldots (1).$$

Now here the powers of x cannot be all different, since the coefficients of them are, either constant quantities not equal to zero, or quantities which we suppose not to approach zero as their limiting value when x approaches zero: for it is clear that if all the powers of x be different, and if the equation be divided by the lowest power, and x then put equal to zero, the coefficient of the lowest power must necessarily become zero, contrary to hypothesis.

Since then the powers of x cannot be all different some two of them must be the same; either the first and second, or the first and third, or the second and third.

(1) Suppose that the first and third are the same; then $3m = m + 1$, and $\therefore$ $m = \frac{1}{2}$; then (1) becomes, dividing out $x^{\frac{3}{2}}$,

$$u^3 - 3au + \text{a positive power of } x = 0,$$

which gives $u = 0$ or $\pm\sqrt{3a}$ when $x = 0$: we must of course reject $u = 0$, since we want only those values of u which are neither 0 nor ∞.

(2) Suppose that $3m = 3$, and $\therefore$ $m = 1$, then we have

$$u^3 + 1 + \text{a negative power of } x = 0,$$

which $\therefore$ gives $u = \infty$ when $x = 0$; this value of m must therefore be rejected.

(3) Suppose that $m + 1 = 3$, and $\therefore m = 2$; then we have

$$-3au + 1 + \text{a positive power of } x = 0,$$

$$\text{which gives } u = \frac{1}{3a} \text{ when } x = 0.$$

Hence it appears that there are two values of m, namely $\frac{1}{2}$ and 2, which answer our purpose, and that when x approaches 0, u in the former case approaches $\sqrt{3a}$ or $-\sqrt{3a}$ as its limiting value, and in the latter case $\frac{1}{3a}$.

Hence, proceeding only as far as the first terms, we have the perfect developements

$$y = \pm\sqrt{3a}\, x^{\frac{1}{2}} + R,$$

$$\text{and } y = \frac{x^2}{3a} + R'.$$

To obtain the second terms of these developements we must assume $m = \frac{1}{2}$, or 2, and

$$u = \pm\sqrt{3a} + v x^n, \quad \text{or } u = \frac{1}{3a} + v x^n \text{ respectively,}$$

and substitute in the equation (1) and proceed as before, and so we may go on to any number of terms. It is often useful to obtain the first terms of the developements as in this Example, but we seldom have occasion to go any farther.

The method stated in general.

It is clear from this Example, that to determine the first terms of the developements, we have only to put $y = a + ux^m$ (a being the value of y when $x = 0$) in the given equation, then suppose the powers of x to be the same, two by two. If, when we suppose two powers the same, any of the remaining powers is lower than them, then there is no value of u such as we want corresponding to these two powers, (as in the case where we assumed $3m = 3$ in the Example). If, however, none of the remaining powers is lower than the two supposed to be the same; then dividing out the lowest power of x from the equation, and putting $x = 0$, we obtain a value or values of u, (rejecting of course any value which = zero); and thus we obtain the developements required.

(For examples and certain simplifications see Appendix I.)

Developement of a function of two variables $f(a+h, b+k)$ in powers of h and k.

††146. *If* f(xy) *be any function of* x *and* y, *and if we put* x = a + h, y = b + k, *to develope* f(xy) *in powers of* h *and* k.

Putting $a + h$ for x, we have, by Taylor's theorem,

$$f(xy) = f(a+h, y) = f(ay) + d_a f(ay)\frac{h}{\Gamma 1} \cdots$$

$$+ d_a^m f(ay) \cdot \frac{h^m}{\Gamma m} + \&c. \ldots (1),$$

but by the same theorem, if we put $b + k$ for y, we have

$$f(ay) = f(a, b+k) = f(ab) + d_b f(ab)\frac{k}{\Gamma 1} \cdots$$

$$+ d_b^n f(ab)\frac{k^n}{\Gamma n} \cdots \&c. \ldots (2).$$

Substituting this value of $f(ay)$ in each term of (1), we obtain the developement required.

Now the general term of (1) is $d_a^m f(ay)\frac{h^m}{\Gamma m}$, and by (2) the general term of this is

$$d_a^m \left\{ d_b^n f(ab)\frac{k^n}{\Gamma n} \right\} \frac{h^m}{\Gamma m}, \quad \text{or} \quad d_a^m d_b^n f(ab) \cdot \frac{h^m k^n}{\Gamma m \Gamma n},$$

hence, giving m and n their several values, we obtain

$$f(a+h, b+k) = f(ab) + d_a f(ab)\frac{h}{\Gamma 1} + d_b f(ab)\frac{k}{\Gamma 1}$$

$$+ d_a^2 f(ab)\frac{h^2}{\Gamma 2} + d_a d_b f(ab)\frac{hk}{\Gamma 1 \Gamma 1} + d_b^2 f(ab)\frac{k^2}{\Gamma 2}$$

$$+ d_a^3 f(ab)\frac{h^3}{\Gamma 3} \ldots\ldots\ldots\ldots\ldots\ldots \&c. \ldots\ldots \&c.$$

$$\ldots\ldots\ldots\ldots \&c. \ldots\ldots\ldots\ldots \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Symbolical form of the developement and of Taylor's Series.

If we use a symbolical form similar to that in 110, this expansion may be expressed in the following manner, {observing that $\frac{\Gamma n}{\Gamma r\,\Gamma(n-r)}$ is the coefficient of $h^r k^{n-r}$ in $(h+k)^n$ expanded}, viz:

$$f(a+h, b+k) = \left\{1 + \frac{(d_a h + d_b k)^1}{\Gamma 1} + \frac{(d_a h + d_b k)^2}{\Gamma 2} \ldots \&c.\right\} f(ab)$$

$$= \epsilon^{d_a h + d_b k} f(ab).$$

Taylor's series expressed similarly may be thus written

$$f(a+h) = \epsilon^{fh}.\,a,$$

$f^0(a)$ being supposed to be the same as $f(a)$.

Another proof of the result that $d_u^m d_v^n y = d_v^n d_u^m y$.

†† 147. In the expansion of $f(a+h,\ b+k)$ above obtained, the coefficient of $\frac{h^m k^n}{\Gamma m\,\Gamma n}$ is $d_a^m d_b^n f(ab)$. Now if we had arrived at this developement by first putting $b+k$ for y and expanding, and then $a+h$ for x and expanding again, it is easy to see that the coefficient of $\frac{h^m k^n}{\Gamma m\,\Gamma n}$ would have been $d_b^n d_a^m f(ab)$, hence we must have

$$d_b^n d_a^m f(ab) = (d_a^m d_b^n f(ab),$$

a result which we obtained before in (106).

CHAPTER X.

DETERMINATION OF THE LIMITING VALUES OF VANISHING FRACTIONS.

Vanishing Fractions.

148. The Differential Calculus may often be employed with great advantage to determine the limiting values of vanishing fractions; i.e. functions which assume the illusory form $\frac{0}{0}$ when the variable receives some particular value.

To determine the limiting value of a vanishing fraction.

If $f(a)=0$ *and* $\phi(a)=0$, *to determine the limiting value of* $\frac{f(x)}{\phi(x)}$ *when* x *approaches* a.

Since $f(a)$ and $\phi(a)$ are zero, $\frac{f(x)}{\phi(x)}=\frac{f(x)-f(a)}{\phi(x)-\phi(a)}$; hence by Lemma XIX. the limiting value of $\frac{f(x)}{\phi(x)}$ is $\frac{f'(a)}{\phi'(a)}$; or more generally, by Lemma XX. the limiting value of $\frac{f'(x)}{\phi'(x)}$ when x approaches a is also the limiting value of $\frac{f(x)}{\phi(x)}$; if it should so happen that $\frac{f'(a)}{\phi'(a)}$ is not an illusory quantity, then it is the limiting value we seek: but if $f'(a)=0$ and $\phi'(a)=0$, then by the same Lemma, Cor. 1, the limiting value of $\frac{f^2(x)}{\phi^2(x)}$ is that of $\frac{f(x)}{\phi(x)}$, and therefore $\frac{f^2(a)}{\phi^2(a)}$, if not illusory, is the value we seek: if however $f^2(a)$ and $\phi^2(a)$ be both zero, then we must try the third differential coefficients, and so on till we come to two differential coefficients which do not both vanish (nor of course become both infinite) when $x=a$; and thus we shall obtain the limiting value we seek.

Case in which this method fails.

If we come to two differential coefficients which both become infinite when $x = a$ then there is no use in going farther, since all the succeeding differential coefficients will be infinite also by Lemma XXIII.

How we are to proceed in such a case.

149. In such a case we must proceed as follows.

Put $a+h$ for x and expand $f(a+h)$ in a perfect series $f(a)+Ah^{\alpha}+R'$ and $\phi(a+h)$ in a perfect series $\phi(a)+Bh^{\beta}+R$; then since $f(a)$ and $\phi(a)$ are each zero, we have

$$\frac{f(a+h)}{\phi(a+h)} = \frac{Ah^{\alpha}+R}{Bh^{\beta}+R'} = h^{\alpha-\beta} \cdot \frac{A+\dfrac{R}{h^{\alpha}}}{B+\dfrac{R'}{h^{\beta}}}.$$

Hence, since the series are perfect and therefore the limiting values of $\dfrac{R}{h^{\alpha}}$ and $\dfrac{R'}{h^{\beta}}$ when h approaches 0 each zero, the limiting value of $\dfrac{f(a+h)}{\phi(a+h)}$ when h approaches zero is the same as that of $h^{\alpha-\beta}\cdot\dfrac{A}{B}$. Now if $\alpha > \beta$ this $= 0$; if $\alpha < \beta$ it $= \infty$; and if $\alpha = \beta$ it $= \dfrac{A}{B}$. Thus the required limiting value is found.

This latter method is sometimes preferable in cases where the former does not fail.

Examples of vanishing fractions.

Let $$\frac{f(x)}{\phi(x)} = \frac{x^n - a^n}{x^3 - ax^2 + a^2x - a^3},$$

which assumes the form $\dfrac{0}{0}$ when $x = a$;

$$\text{here } f'(x) = nx^{n-1} = na^{n-1} \text{ when } x = a,$$
$$\phi'(x) = 3x^2 - 2ax + a^2 = 2a^2 \text{ when } x = a;$$

hence the limiting value of $\dfrac{f(x)}{\phi(x)}$ when x approaches a is $\dfrac{f'(a)}{\phi'(a)}$ or $\dfrac{n}{2}a^{n-3}$.

Example 2.

Let $\dfrac{f(x)}{\phi(x)} = \dfrac{\tan x - \sin x}{x^3}$, which assumes the form $\dfrac{0}{0}$ when $x = 0$,

$$\frac{f'(x)}{\phi'(x)} = \frac{\sec^2 x - \cos x}{3x^2} = \frac{0}{0} \text{ when } x = 0$$

$$= \frac{1}{\cos^2 x} \cdot \frac{1 - \cos^3 x}{3x^2}.$$

Important simplification we may often make in the process.

150. Here we must make a remark of some importance. Suppose that we obtain $\dfrac{f'(x)}{\phi'(x)}$ in the form UV, and that we know A to be the limiting or actual value of U corresponding to the value of x which makes the fraction vanish; then, by Lemma VIII., the limiting value we require is A × limiting value of V. Hence if in the process of obtaining the limiting value of a vanishing fraction, we find that $\dfrac{f'(x)}{\phi'(x)}$ has any factor whose limiting or actual value we know, we may always substitute immediately for that factor its value whatever it be, and so simplify the operation.

Thus in the present example the factor $\dfrac{1}{\cos^2 x} = 1$ when $x = 0$, we may therefore substitute 1 for it, and then we have only to find the limiting value of $\dfrac{1 - \cos^3 x}{3x^2}$; i.e. of

$$\frac{d(1 - \cos^3 x)}{d(3x^2)} \text{ or } \frac{\cos^2 x \sin x}{2x},$$

and here again putting 1 instead of the factor $\cos^2 x$ we have only to find the limiting value of $\dfrac{\sin x}{2x}$; i.e. of $\dfrac{d \sin x}{d\,2x}$ or $\dfrac{\cos x}{2}$, which is $\frac{1}{2}$. Hence the limiting value required is $\frac{1}{2}$.

Example 3. Let $\frac{f(x)}{\phi(x)} = a^{n-1}.\frac{a-(a^n-x^n)^{\frac{1}{n}}}{x^n}$, which assumes the form $\frac{0}{0}$ when $x=0$,

$$\frac{f'(x)}{\phi'(x)} = a^{n-1}.\frac{x^{n-1}(a^n-x^n)^{\frac{1}{n}-1}}{n x^{n-1}} = \frac{a^{n-1}}{n}(a^n-x^n)^{\frac{1}{n}-1},$$

the limiting value of which when $x=0$ is evidently $\frac{1}{n}$.

Example 4. Let $\frac{f(x)}{\phi(x)} = (1-x)\tan\frac{\pi x}{2}$, which assumes the illusory form $0.\infty$ when $x=1$. This is the same thing as the form $\frac{0}{0}$, as appears if we put $\frac{f(x)}{\phi(x)}$ in the form

$$\frac{(1-x)\sin\frac{\pi x}{2}}{\cos\frac{\pi x}{2}}.$$

Now the limiting value of this is the same as that of

$$\frac{1-x}{\cos\frac{\pi x}{2}} \text{ (since } \sin\frac{\pi x}{2} = 1 \text{ when } x=1\text{);}$$

$$\text{i. e. of } \frac{-1}{-\frac{\pi}{2}\sin\frac{\pi x}{2}} \text{ which } = \frac{2}{\pi}.$$

The illusory form $\infty-\infty$ is the same as the form $\frac{0}{0}$.

151. A function sometimes assumes the form $\infty-\infty$, which is the same form as $\frac{0}{0}$: for let $\frac{1}{f(x)} - \frac{1}{\phi(x)}$ be a function which assumes the form $\infty-\infty$ when $x=a$, in consequence of $f(a)$ and $\phi(a)$ being each zero: then

$$\frac{1}{f(x)} - \frac{1}{\phi(x)} = \frac{\phi(x)-f(x)}{f(x)\,\phi(x)},$$

which assumes the form $\frac{0}{0}$ when $x = 0$; thus the illusory forms $\frac{0}{0}$ and $\infty - \infty$ are identical.

How to find the limiting value of a function when it becomes $\infty - \infty$.

Hence when we wish to find the limiting value of a function which assumes the form $\infty - \infty$, we must reduce it to a fraction, so that it shall assume the form $\frac{0}{0}$, and then proceed as above.

Example 5.

$\frac{2}{x^2 - 1} - \frac{1}{x - 1}$ becomes $\infty - \infty$ when $x = 1$; to find its limiting value we must put it in the form $\frac{2 - (x + 1)}{x^2 - 1}$, which assumes the form $\frac{0}{0}$, and whose limiting value therefore is that of $\frac{-1}{2x}$, or $-\frac{1}{2}$.

Example 6. Case of failure of common method.

152. Let $\frac{f(x)}{\phi(x)} = \frac{x - a + \sqrt{2ax - 2a^2}}{\sqrt{x^2 - a^2}}$, which assumes the form $\frac{0}{0}$ when $x = a$.

Here $f'(a)$ and $\phi'(a)$ are both infinite, we must therefore put $x = a + h$, and then $\frac{f(x)}{\phi(x)}$ becomes $\frac{h + \sqrt{2ah}}{\sqrt{h(2a + h)}}$, which $= \frac{h^{\frac{1}{2}} + \sqrt{2a}}{\sqrt{2a + h}}$, which $= 1$ when $h = 0$, hence the required limiting value is 1.

Example 7. A similar case.

$$\text{Let } \frac{f(x)}{\phi(x)} = \frac{\sin^3 \sqrt{x^2 - 1} + (x - 1)^2}{(x^2 - 1)\sqrt{x - 1}},$$

which assumes the form $\frac{0}{0}$ when $x = 1$; here $f^2(1)$ and $\phi^2(1)$

are both infinite; we must therefore put $x = 1 + h$, and then $\frac{f(x)}{\phi(x)}$ becomes

$$\frac{\sin^3 \sqrt{h(h+2)} + h^2}{h(h+2)h^{\frac{1}{2}}}.$$

Now $\sin z = z - \frac{z^3}{6}$ &c....

$$\sin^3 z = z^3 - \frac{z^5}{2} \text{ \&c....}$$

$\therefore$ putting $z^3 = [\{h(h+2)\}]^{\frac{3}{2}} = 2^{\frac{3}{2}}h^{\frac{3}{2}} +$ &c....

we find $f(x) = h^{\frac{3}{2}} + R$ a perfect developement,

and $\phi(x) = 2h^{\frac{3}{2}} + R'$ ditto,

and $\therefore$ the required limiting value is that of $\frac{2^{\frac{3}{2}}h^{\frac{3}{2}}}{2h^{\frac{3}{2}}}$ which $= \sqrt{2}$. (See Appendix J.)

When we are finding $\frac{dy}{dx}$ from an equation it sometimes assumes the form $\frac{0}{0}$.

††153. It often happens, when we are finding $\frac{dy}{dx}$ from an equation between y and x, as in 145, that the result comes out in the form $\frac{0}{0}$, which of course leaves us in ignorance as to what the true value of $\frac{dy}{dx}$ is. Thus suppose that we wish to obtain the value of $\frac{dy}{dx}$ when $x = 0$ from the equation,

$$x^4 + 3a^2x^2 - 4a^2xy - a^2y^2 \text{ .... } (1).$$

Differentiating, we have

$$4x^3 + 6a^2x - 4a^2y - (4a^2x + 2a^2y)\,p = 0 \text{ ...... } (2) \left(p = \frac{dy}{dx}\right).$$

Now putting $x = 0$, and $\therefore y = 0$ in virtue of (1), we have

$$0 - 0 \,.\, p = 0, \text{ or } p = \frac{0}{0};$$

thus when $x = 0$ we cannot find p in this manner.

In such a case we must differentiate again in order to find $\frac{dy}{dx}$.

But differentiating the equation (2) we have

$$6x^2 + 3a^2 - 2a^2p - (2a^2 + a^2p)\,p - (2a^2x + a^2y)\,q = 0 \ \left(q = \frac{d^2y}{dx^2},\right.$$

and putting $x = 0$, and $\therefore\ y = 0$, here we find

$$3a^2 - 2a^2p - (2a^2 + a^2p)\,p = 0,$$

$$\text{or } p^2 + 4p - 3 = 0;$$

$$\therefore\ p = -2 \pm \sqrt{4+3},$$

and thus by differentiating twice we obtain p, and we find that it has two values $-2 + \sqrt{7}$ and $-2 - \sqrt{7}$.

Another example.

As another example, suppose that we wish to obtain p when $x = 0$ from the equation

$$x^4 + ay^3 - 2axy^2 - 3ax^2y = 0 \ldots\ldots\ldots (1),$$

differentiating we find

$$(4x^3 - 2ay^2 - 6axy) + (3ay^2 - 4axy - 3ax^2)\,p = 0 \ldots\ldots (2),$$

here put $x = 0$, and $\therefore\ y = 0$ in virtue of (1), and we find $p = \frac{0}{0}$.

But differentiating again we have

$$\frac{12x^2}{a} - 6y - 4(2y + 3x)\,p + 2(3y - 2x)\,p^2 + (3y^2 - 4xy - 3x^2)\,q = 0,$$

here put $x = 0$, and $\therefore\ y = 0$, and we find again $p = \frac{0}{0}$.

Differentiating again, therefore, we have

$$0 = -18p - 12p^2 + 6p^3 + \text{terms multiplied by } x \text{ or } y,$$

here putting $x = 0$ and $y = 0$, we find

$$p^3 - 2p^2 - 3p = 0,$$

which gives $p = 0$ or $p = 1 \pm \sqrt{1+3} = 3$ or -1.

And thus by differentiating three times we obtain p, and we find that it has three values 0, 3 and -1.

Thus it appears that where the first differentiation of an equation fails to give us $\frac{dy}{dx}$ for a particular value of x, in consequence of all the terms becoming zero, we must differentiate successively until we come to an equation, all the terms of which do not vanish when we give x that particular value.

These differentiations may be performed on the supposition that p is constant.

††154. If we perform these differentiations on the supposition that p is a constant, we shall arrive at a correct result: for it is easy to see from the above examples that all the terms obtained by differentiating p once or oftener vanish when we give x the particular value for which we wish to find p, and therefore do not affect our final equation, which gives us the value of p. Hence we may always differentiate on the supposition that p is constant, and this will somewhat simplify the process.

A somewhat dfferent method of finding p by the rule for vanishing fractions.

††155. The values of p are sometimes found by the method of finding the limiting value of a vanishing fraction given in (148); thus in the first example we have

$$p = \frac{2x^3 + 3a^2x - 2a^2y}{2a^2x + a^2y} = \frac{0}{0} \text{ when } x = 0,$$

therefore by (148),

$$p = \frac{6x^2 + 3a^2 - 2a^2p}{2a^2 + a^2p} = \frac{3 - 2p}{2 + p} \text{ when } x = 0;$$

$$\therefore\ 2p + p^2 = 3 - 2p, \quad p^2 + 4p - 3 = 0,$$

which is the result we arrived at before.

Another example.

†† 156. The following is an example where p assumes the form $\frac{0}{0}$ for other values of x and y besides zero.

To find p when $x = a$ from the following equation, viz.

$$ay^2 - 2a^2y - 2x^3 + 3ax^2 = 0 \ldots\ldots (1).$$

Differentiating, we have

$$(2ay - 2a^2)\,p - 6x^2 + 6ax = 0.$$

When $x = a$, and therefore $y = a$, in virtue of (1), this equation gives $p = \frac{0}{0}$; therefore, differentiating again, considering p constant, we have

$$2ap^2 - 12x + 6a = 0;$$

which, when $x = a$, gives

$$p^2 - 3 = 0; \quad \therefore \; p = \pm\sqrt{3}.$$

Objections to this method of finding p by successive differentiation.

†† 157. There are two objections to this method of finding the values of p. 1st. It is generally very complicated and troublesome when we have to go beyond a second differentiation. 2nd. We have no right to assume that the terms containing $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ ... vanish; for although the coefficients of these terms vanish, yet, since $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ ... may and often do at the same time become infinite, we cannot tell but that these terms may assume the form $0 . \infty$ or $\frac{0}{0}$; and therefore we cannot assume them to be zero. For instance, in the first example, (153), in the result of the second differentiation we have the term $(2a^2x + a^2y)\,q$, which we assume to be zero because $2ax + a^2y = 0$ when $x = 0$; but we have no right to do this, since we cannot tell whether q is infinite or not.

A method free from these objections explained.

†† 158. The following method will be found free from this objection, and very simple in practice, especially when it is our object to find p for the values $x = 0$ and $y = 0$.

(1) Suppose that we wish to find p for the value $x = 0$ from the equation

$$Ax + By + Cx^2 + Dxy + Ey^2 + Fx^3 \dots \text{\&c.} = 0 \;\dots\dots\; (1).$$

Since $y = 0$ when $x = 0$, it is evident that the value of p we seek is the limiting value of $\frac{y}{x}$ when x approaches zero (see 148): if therefore we put $\frac{y}{x} = u$ and find the limiting value of u, u_0 suppose, when x approaches zero, u_0 is the quantity we wish to determine.

Now put $y = ux$ in (1), divide out x, and we find

$$A + Bu + x(C + Du + Eu^2) + x^2(F \ldots) \ldots\ldots (2),$$

here let $x = 0$, and then u must become u_0, if therefore u_0 be a finite quantity we have

$$(3) \ldots. A + Bu_0 = 0, \text{ and } \therefore u_0 = -\frac{A}{B};$$

which gives u_0.

If it should happen, however, that both A and B are zero, the value we have here obtained for u_0 assumes the form $\frac{0}{0}$, and therefore we cannot thus arrive at the real value of u_0. But then the equation (2), dividing out x, becomes

$$C + Du + Eu^2 + x(F + \&c.\ldots) \ldots = 0;$$

here put $x = 0$, and therefore $u = u_0$, and we find (supposing u_0 not infinite),

$$C + Du_0 + Eu_0^2 = 0;$$

$$\text{and } \therefore u_0 = -\frac{D}{2E} \pm \sqrt{\frac{D^2}{4E^2} - \frac{C}{4E}};$$

thus it appears that $\frac{dy}{dx}$ has two values in this case, when $x = 0$.

How the infinite values of u_0 are to be found.

In this process we suppose u_0 to be a finite quantity; therefore, if u_0 admits of infinite values, this method does not give them, and we are left in ignorance as to whether there are such values or not*. But we may easily determine this in the following manner:

Put $x = \frac{y}{u}$ instead of $y = ux$, and the equation (1) becomes (supposing A and B zero)

$$C\frac{1}{u^2} + D\frac{1}{u} + E + y\left(F\frac{1}{u^3} \ldots \&c.\right), \&c. = 0;$$

---

[*] In the common method of finding $\frac{dy}{dx}$ the infinite values are sometimes overlooked.

and then putting $y = 0$, we have

$$C\frac{1}{u_0^2} + D\frac{1}{u_0} + E = 0.$$

Hence if E is not zero $\frac{1}{u_0}$ cannot be zero; and therefore u_0 cannot be infinite: but if $E = 0$, then one value of $\frac{1}{u_0}$ is zero, and therefore one value of u_0 infinite: if D also $= 0$, both values of $\frac{1}{u_0}$ are zero, and therefore both values of u_0 infinite.

If C, D, and E be all zero, then in the equation (2), dividing out x^2, and putting $x = 0$, we have

$$F + Gu_0 + Hu_0^2 + Iu_0^3 = 0 \; \ldots.$$

which equation gives us the finite values of u_0. To obtain the infinite values we have, as before, the equation

$$F\frac{1}{u_0^3} + G\frac{1}{u_0^2} + H\frac{1}{u_0} + I = 0.$$

It appears from these equations that there are in this case 3 values of $\frac{dy}{dx}$ when $x = 0$. If $I = 0$ one of the values of $\frac{1}{u_0}$ is zero, and therefore one of the values of u_0 is infinite: if $I = 0$, $H = 0$, two of the values of $\frac{1}{u_0}$ are zero, and therefore two of u_0 are infinite: if $G = 0$ also, the three values of u_0 are infinite: if $I = 0$ and $F = 0$, a value of u_0 is 0, and a value of $\frac{1}{u_0}$ is 0, and either of the equations becomes $G + Hu_0 = 0$, which gives $u_0 = -\frac{G}{H}$; hence, in this case, the three values of u_0 are 0, ∞, and $-\frac{G}{H}$.

If F, G, H, I, are all zero, we must divide out x^3 in equation (2) and proceed as before; and thus we may in all cases obtain an equation for determining all the values of u_0.

Rule for detecting the infinite values of u_0 immediately.

If in general we divide out x^n and the resulting equation for determining u_0 is only of the $n-m^{\text{th}}$ order, then m of the values of u_0 are infinite, as is easily seen. By remembering this we may detect the infinite values of u_0 immediately.

Example 1.

†† 159. Let the given equation be (see 153)

$$x^4 + 3a^2x^2 - 4a^2xy - a^2y^2 = 0,$$

here putting $y = ux$, dividing out x^2 and putting $x = 0$, we have

$$3a^2 - 4a^2u_0 - a^2u_0^{\,2} = 0, \quad \text{or } u_0^{\,2} - 4u_0 - 3 = 0,$$

which gives us the same values of $\dfrac{dy}{dx}$ that we obtained before.

Example 2.

Let the given equation be (see 153)

$$x^4 + ay^3 - 2axy^2 - 3ax^2y = 0,$$

here putting $y = ux$, dividing out x^3 and putting $x = 0$, we find

$$au_0^{\,3} - 2au_0^{\,2} - 3au_0 = 0,$$

$$\text{or } u_0^{\,3} - 2u_0^{\,2} - 3u_0 = 0,$$

which gives us the same values we obtained before of $\dfrac{dy}{dx}$.

Example 3.

Let the given equation be

$$y^5 + ax^4 - b^2xy^2 = 0,$$

in this case we find, putting $y = ux$ and dividing out x^3,

$$u_0^{\,2} = 0,$$

hence, and therefore the three values of u_0 are 0, 0, and ∞.

How to proceed when the values of x and y which make $\dfrac{dy}{dx}$

†† 160. We have hitherto supposed that the values of x and y which make $\dfrac{dy}{dx}$ assume the form $\dfrac{0}{0}$ are $x = 0$ and $y = 0$: but let us now suppose them to be $x = a$ and $y = b$,

assume the form $\frac{0}{0}$ are not zero.

then putting $x = a + x_{\prime}$ and $y = b + y_{\prime}$ in the given equation $\frac{dy_{\prime}}{dx_{\prime}}$ will be the same thing as $\frac{dy}{dx}$, and $x_{\prime} = 0$ $y_{\prime} = 0$ will be the values of $x_{\prime}$ and $y_{\prime}$ which make $\frac{dy_{\prime}}{dx_{\prime}}$ assume the form $\frac{0}{0}$; we may therefore proceed as before. If the given equation be

$$U = 0,$$

and if we denote by A, B, C, D, E... &c. the values of $d_x U$, $d_y U$, $d_x^2 U$, $2 d_x d_y U$, $d_y^2 U$, &c. when x is put $= a$ and $\therefore\ y = b$; then, by 146, the result of substituting $a + x_{\prime}$ for x and $b + y_{\prime}$ for b will be

$$Ax_{\prime} + By_{\prime} + Cx_{\prime}^2 + Dx_{\prime}y_{\prime} + Ey_{\prime}^2 + \&c.\ldots = 0;$$

if therefore A and B be not both zero, we have

$$A + Bu_0 = 0,$$

and if A and B be both zero we have,

$$C + Du_0 + Eu_0^2 = 0,$$

....

and so on...........

and we may find u_0 therefore just as before.

Example.

Let the given equation be (see 156)

$$U = ay^2 - 2a^2y - 2x^3 + 3ax^2 = 0,$$

from which we wish to find $\frac{dy}{dx}$ when $x = a$, and $\therefore\ y = b$,

$$\text{then } d_x U = -6x^2 + 6ax = 0 \text{ when } x = a,$$

$$d_y U = 2ay - 2a^2 = 0 \text{ when } y = a,$$

$$d_x^2 U = -12x + 6a = -6a \text{ when } x = a,$$

$$d_x d_y U = 0,$$

$$d_y^2 U = 2a.$$

Hence $A = 0$, $B = 0$, $C = -6a$, $D = 0$, $E = 2a$, and therefore we have

$$-3 + u_0^2 = 0, \text{ and } \therefore\ u_0 = \pm\sqrt{3} \text{ as before.}$$

CHAPTER XI.

DETERMINATION OF THE MAXIMA AND MINIMA VALUES OF FUNCTIONS.

Maxima and Minima values defined.

161. THE Differential Calculus may be applied with great success to determine the maxima and minima values of a function, i.e. those particular values which are either greater or less than any of the neighbouring values.

If $f(x)$ increases when x approaches a certain value a (supposing x to increase continually), and diminishes when x passes the value a, then $f(a)$ must be greater than any value of $f(x)$ which is either a little less or a little greater than a, and is therefore called a maximum value of $f(x)$. And if $f(x)$ diminishes when x approaches a and increases when x passes a, $f(a)$ must be less than any value of $f(x)$ which is either a little less or a little greater than a, and is therefore called a minimum value of $f(x)$.

How these values of $f(x)$ may be found by means of $f(x)$.

162. Now by Lemma XVIII. $f(x)$ is increasing or diminishing according as $f'(x)$ is positive or negative: therefore if $f(a)$ be a maximum, $f'(x)$ must be positive for all values of x a little less than a, and negative for all values of x a little greater than a; i.e. $f'(x)$ must change its sign from $+$ to $-$ when x passes through the value a: and, conversely, if $f'(x)$ changes its sign from $+$ to $-$ when x passes through the value a, $f(a)$ is a maximum.

In like manner if $f(a)$ be a minimum, $f'(x)$ must change its sign from $-$ to $+$ when x passes through the value a; and conversely if $f'(x)$ does so change its sign, $f(a)$ is a minimum.

Hence by Lemma XVII if $f(a)$ be a maximum or minimum, $f'(a)$ must be zero or infinity: but of course, since $f'(a)$ may be zero or infinity without $f(x)$ changing its sign, it does not follow that $f(a)$ must be a maximum or minimum whenever $f'(a)$ is zero or infinity.

Rule for determining maxima and minima.

163. *Hence to determine the maxima and minima of* $\mathrm{f}(\mathrm{x})$, *we must determine what values of* x *make* $\mathrm{f}'(\mathrm{x})$ *zero or infinity, and try whether* $\mathrm{f}'(\mathrm{x})$ *changes its sign when* x *passes through each of these values; those values which give a change from* + *to* − *make* $\mathrm{f}(\mathrm{x})$ *a maximum; those which give a change from* − *to* + *make* $\mathrm{f}(\mathrm{x})$ *a minimum; and those which do not give a change must be rejected.*

Factors of $f'(x)$ may be suppressed under certain circumstances

164. In applying this method to any example, we may suppress any factor of $f'(x)$ which we are sure is always positive, or may introduce any such factor; since we are not concerned with the actual magnitude of $f'(x)$ but only its sign. This will often considerably simplify our operations, as will appear.

If therefore we find $f'(x)$ in the form $\phi(x) \cdot \psi(x)$, and if $\phi(x)$ be always a positive quantity, then we may put $f'(x) = \psi(x)$ simply.

If $\phi(x)$ be always a negative quantity, then $f'(x)$ will have the same sign as $-\psi(x)$, and therefore we may put $f'(x) = -\psi(x)$.

165. If it be our object simply to examine whether $f'(x)$ changes its sign, and how, when x passes through a certain value a, then if $\phi(a)$ be neither zero nor infinity, we may suppose $f'(x) = +\psi(x)$ if $\phi(a)$ be positive, and $f'(x) = -\psi(x)$ if $\phi(a)$ be negative.

For it is clear that if $\phi(a)$ be neither zero nor infinity, then for all values of x near a $\phi(x)$ has the same sign as $\phi(a)$, and therefore $\phi(x) \cdot \psi(x)$ or $f'(x)$ the same sign as $\phi(a)\ \psi(x)$, i.e. $f'(x)$ has the same sign as $+\psi(x)$ or $-\psi(x)$ according as the sign of $\phi(a)$ is + or − for all values of x near a, and therefore in examining whether $f'(x)$ changes its sign, and how, when x passes through a, we may put $f'(x) = +\psi(x)$ or $-\psi(x)$ according as $\phi(a)$ is positive or negative.

We may therefore suppress all factors of $f'(x)$ which do not become 0 or ∞ when $x = a$, retaining only the signs they have when $x = a$. This is a very important simplification.

$\phi(u)$ is a maximum or minimum whenever u is, if $\phi(u)$ always increases when u increases: if not the reverse is the case.

166. If $\phi(u)$ always increases when u increases, it is a maximum or minimum whenever u is a maximum or minimum respectively; for $\dfrac{d\phi(u)}{dx} = \phi'(u)\dfrac{du}{dx}$: and since $\phi(u)$ always increases when u increases, $\phi'(u)$ is always positive, and therefore $\dfrac{d\phi(u)}{dx}$ has always the same sign as $\dfrac{du}{dx}$, and $\therefore$ $\phi(u)$ and u become maxima or minima at the same time. If $\phi(u)$ diminishes when x increases the reverse is the case. We often find this consideration of use in practice, inasmuch as it may in many cases be much easier to find the maxima and minima of $\phi(u)$ than of u.

We may determine whether and how $f'(x)$ changes its sign when x passes through a, by the consideration of the second or higher differential coefficient.

167. If $f'(a) = 0$, then by Lemma XX, Cor. 4, $f'(x)$ has in general the same sign as $f^2(a)(x - a)$ for all values of x taken sufficiently near a: therefore if $f^2(a)$ be positive $f'(x)$ changes its sign from $-$ to $+$ when x passes through the value a, which indicates a maximum; and if $f^2(a)$ be negative the change is from $+$ to $-$, which indicates a maximum. If however $f^2(a) = 0$, let $f^n(a)$ be the first differential coefficient of $f(x)$, which does not vanish when $x = a$; then $f'(x)$, (by Lemma XX, Cor. 4), has the same sign as $f^n(a)(x - a)^{n-1}$ for all values of x sufficiently near a. If, therefore, n be odd, $f'(x)$ does not change its sign when x passes through the value n; but if n be even it does, and the change is from $-$ to $+$ or from $+$ to $-$ according as $f^n(a)$ is positive or negative. Of course we here suppose that none of the differential coefficients are infinite.

Simple inspection is often the best way and sometimes the only way of making out whether $f'(x)$ changes its sign and how.

These considerations will enable us to determine whether $f'(x)$ changes its sign when x passes through the value a, and if so, whether the change is from $-$ to $+$ or from $+$ to $-$. But this is often more easily seen by simple inspection; and indeed when $f'(a)$ or any of the higher differential coefficients are infinite, which often occurs, simple inspection is the only method we can resort to.

By suppressing factors of $f'(x)$ which do not change their sign (in the manner shewn to be allowable in 164, 165,) before we differentiate $f'(x)$, we may often find the sign of $f^2(a)$, or more properly, of that quantity which answers the same purpose as $f^2(a)$, with great facility, *and avoid the necessity of being obliged to proceed to higher differential coefficients.*

When we obtain $y-b$ in the form $A(x-a)^m + R$, we may immediately see whether $x=a$ makes y a maximum or minimum.

168. If $y=b$ when $x=a$, and we can obtain a perfect developement in the form

$$y - b = A(x-a)^m + R = A(x-a)^m \left\{1 + \frac{R}{(x-a)^m}\right\},$$

then by taking x near enough to a, we may make $\frac{R}{(x-a)^m}$ as small as we please, and then $y-b$ will have the same sign as $A(x-a)^m$. Therefore if m be an odd number or a fraction with an odd numerator and an odd denominator and A positive, y is $<b$ when x is $<a$, and $>b$ when x is $>a$, and *vice versâ* if A be negative: in this case therefore b is not a maximum nor minimum value of y. But if m be an even number or a fraction with an even numerator and odd denominator and A positive, then y is $>b$ for all values of x near a, and therefore b is a minimum value of y: and if A be negative then y is less than b for all values of x near a, and therefore b is a maximum value of y. If m be a fraction in its lowest terms with an even denominator, then y is impossible for all values of x less than a $\{$or for all values of x greater than a if $y - b = A(a-x)^m + R\}$, in which case we cannot call b a maximum or minimum value of y.

169. The following examples will shew the advantages of the method of finding maxima and minima here recommended.

Example 1.

Let $$f(x) = x^5 (x-a)^3,$$

then $$f'(x) = x^4 (x-a)^2 (8x - 5a)$$

$$= 8x - 5a,$$

suppressing the factor $x^4 (x-a)^2$ which is always positive see (164).

Now $8x - 5a = 0$ when $x = \dfrac{5a}{8}$, and changes its sign from $-$ to $+$ when x passes through the value $\dfrac{5a}{8}$: hence $x = \dfrac{5a}{8}$ gives a minimum value of $f(x)$, viz.

$$\left(\frac{5a}{8}\right)^5 \left(\frac{5a}{8} - a\right)^3 = -5^5 3^3 \left(\frac{a}{8}\right)^8.$$

Or thus $\quad f'(x) = 2x - 5a$;

$$\therefore\ f^2(x) = 2;\quad \therefore\ f^2\left(\frac{5a}{8}\right) = \text{positive},$$

which indicates a minimum by (167).

Example 2. Let $\quad f(x) = \dfrac{x^3}{(x-a)^5},$

then $\quad f'(x) = \dfrac{x^2}{(x-a)^6}(-2x - 3a)$

$$= -(2x + 3a), \text{ suppressing the factor } \frac{x^2}{(x-a)^6}.$$

Now $-(2x + 3a)$ is zero when $x = -\dfrac{3a}{2}$, and changes its sign from $+$ to $-$ when x passes through that value: hence $x = -\dfrac{3a}{2}$ gives a maximum value of $f(x)$.

Or thus, $\quad f'(x) = -(2x + 3a)$

$$f^2(x) = -2;\quad \therefore\ f\left(-\frac{3a}{2}\right) = \text{negative},$$

which indicates a maximum.

Example 3. Let $\quad f(x) = \dfrac{\sin mx}{\sin x}$, m being an integer,

then $\quad f'(x) = \dfrac{m\cos mx \sin x - \sin mx \cos x}{\sin^2 x}$

$$= m\cos mx \sin x - \sin mx \cos x \ \ldots\ldots\ (1),$$

suppressing the factor $\frac{1}{\sin^2 x}$,

$$= \cos mx \cos x\,(m \tan x - \tan mx).$$

Now it is clear from geometrical considerations that there is some value of mx between 0 and $\frac{\pi}{2}$, which makes $\tan mx = m \tan x$: let this value be ma; then $f'(a) = 0$. Also differentiating (1),

$$f^2(x) = -(m^2 - 1)\sin mx \sin x,$$

which is negative when $x = a$, since m^2 is > 1, and ma and therefore a between 0 and $\frac{\pi}{2}$. Hence the root of the equation $\tan mx - m \tan x = 0$ which multiplied by m lies between 0 and $\frac{\pi}{2}$ makes $\frac{\sin mx}{\sin x}$ a maximum.

Example 4.

Let $f(x) = \left(\frac{\sin mx}{\sin x}\right)^2$,

$$\text{then}\quad f'(x) = 2\frac{\sin mx}{\sin^3 x}(m \cos mx \sin x - \sin mx \cos x)$$

$$= \sin mx \sin x\,(m \cos mx \sin x - \sin mx \cos x),$$

multiplying by the factor $\frac{\sin^4 x}{2}$.

Now $x = a$ (see last example) makes $f'(x) = 0$; also $\sin mx \sin x$ is positive when $x = a$, and may therefore be suppressed so far as this value of x is concerned, therefore $f^2(x)$ has the same value as before and is therefore negative when $x = a$; which indicates a maximum.

Again, $x = \frac{\pi}{m}$ makes $\sin mx = 0$, and the product of the other factors of $f'(x)$ negative; therefore by (165) so far as this value of x is concerned we may put

$$f'(x) = -\sin mx;$$

$$\therefore\ f^2(x) = -m\cos mx = \text{positive when } x = \frac{\pi}{m},$$

hence $x = \frac{\pi}{m}$ gives a minimum.

Example 5. Let

$$f(x) = (x-a)^{\frac{2}{3}},$$

$$\text{then}\quad f'(x) = \tfrac{2}{3}(x-a)^{-\frac{1}{3}}$$

$$= x - a,$$

multiplying by the factor $\frac{3}{2}(x-a)^{\frac{4}{3}}$ which is always positive;

$$\therefore\ f^2(x) = 1,$$

hence $f'(a) = 0$, and $f^2(a) =$ positive; and therefore $x = a$ gives a minimum.

(For more Examples see Appendix K.)

Maxima and minima of functions of several variables.

†† 170. Let $f(xy)$ be a function of two independant variables x and y, and let $f(x_1 y)$ be a maximum value of $f(xy)$, determined on the supposition that y is constant and x alone variable, x_1 of course being some function of y; and again let $f(ab)$ be a maximum value of $f(x_1 y)$ determined on the supposition that y is variable, a being the value of x_1 when y becomes b: then $f(ab)$ is a maximum value of $f(xy)$ when x and y are both supposed to vary in any manner.

For supposing x to have any value near x_1, $f(x_1 y)$ is $> f(xy)$; and supposing y to have any value near b, and therefore x_1 some value near a, $f(ab)$ is $> f(x_1 y)$, and therefore *à fortiori* $f(ab)$ is $> f(xy)$. Hence if xy have any values near a and b, $f(ab)$ is $> f(xy)$, and therefore $f(ab)$ is a maximum value of $f(xy)$, supposing x and y both to vary in any manner.

Partial and total maxima and minima.

We may call $f(ab)$ a *total* maximum in contradistinction to $f(x_1 y)$, which is only a *partial* maximum determined on the supposition that only x varies.

In the same manner if $f(x_1 y)$ be a *partial* minimum value of $f(xy)$, y being considered constant, and $f(ab)$ a minimum value of $f(x_1 y)$, y being considered variable; then $f(ab)$ is a *total* minimum value of $f(xy)$.

Hence a maximum value of a partial maximum is a total maximum, and a minimum of a partial minimum is a total minimum.

Conversely, if $f(ab)$ be a total maximum, it must be the maximum of a partial maximum; for $f(ab)$ must be $> f(xy)$ for all values of x and y near a and b; therefore $f(ab)$ must be $> f(x_1 y)$, y being supposed to have any value near b, and therefore x some value near a; therefore $f(ab)$ is a maximum value of $f(x_1 y)$, i.e. it is a maximum of a partial maximum.

And similarly, if $f(ab)$ be a total minimum, it must be a minimum of a partial minimum.

Hence by finding the maxima of the partial maxima of $f(xy)$, and the minima of the partial minima, we find *all* the total maxima and minima values of $f(xy)$.

And thus, by the methods already given of finding maxima and minima of functions of one variable, we may obtain the total maxima and minima of functions of two independant variables.

Example.

Let $f(xy) = x^4 + y^4 - 4axy^2$,

then to find the partial maxima and minima of $f(xy)$ on the supposition that x is variable, we have

$$d_x f(xy) = 4x^3 - 4ay^2,$$

which $= 0$ when $x = (ay^2)^{\frac{1}{3}}$, and evidently changes its sign from $-$ to $+$ when x passes through this value; which indicates a minimum. Therefore substituting $x = (ay^2)^{\frac{1}{3}}$ in $f(xy)$ we obtain the partial minimum

$$y^4 - 3a^{\frac{4}{3}}y^{\frac{8}{3}} = \psi(y) \text{ suppose.}$$

Then to find the minimum of this, we have

$$\psi'(y) = 4y^{\frac{5}{3}}(y^{\frac{4}{3}} - 2a^{\frac{4}{3}}),$$

which $= 0$ when $y = 0$ or $2^{\frac{3}{4}}a$: when y passes through the former value $\psi'(y)$ changes its sign from + to −, and through the latter from − to +: the latter therefore makes $\psi'(y)$ a minimum.

Hence $f(xy)$ has a total minimum value obtained by putting $x = (ay^2)^{\frac{1}{3}}$ and $y = 2^{\frac{3}{4}}a$; i. e. $x = 2^{\frac{1}{2}}a$ and $y = 2^{\frac{3}{4}}a$.

(For more Examples see Appendix K.)

The equations $d_x f(xy) = 0$ or ∞, $d_y f(xy) = 0$ or ∞, give us those values of x and y, which may make $f(xy)$ a total maximum or minimum.

†† 171. Since $f(x_1 y)$ is a maximum or minimum on the supposition that y is constant, it is clear that $d_x f(xy)$ must become 0 or ∞ when x_1 is put for x, and this being true whatever value is assigned to y, it must be true when $y = b$ and therefore $x = a$. Hence if $f(ab)$ be a total maximum or minimum, we must have

$$d_x f(xy) = 0 \text{ or } \infty \ \dots \ (1)$$

when a and b are put for x and y.

And in exactly the same way we may shew, by treating y as we have done x, and x as we have done y, that we must have

$$d_y f(xy) = 0 \text{ or } \infty \ \dots \ (2)$$

when a and b are put for x and y.

a and b are therefore values of x and y got from the equations (1) and (2) taken together; and these equations therefore give us all the values of x and y which *may* make $f(xy)$ a total maximum or minimum.

The method of distinguishing maxima and minima of functions of two variables given by Lagrange is often troublesome; and since it does not include the cases where any of the partial differential coefficients of $f(xy)$ become infinite, or where those of the second order vanish, it must be considered as very incomplete.

How to find maximum and minimum values of y, when

†† 172. When y and x are connected by an equation, and we wish to find what values of x make y a maximum or minimum, we have only to differentiate the equation to find

$\frac{dy}{dx}$ and proceed as before. The following example will explain the process.

an equation is given between x and y.

Given $y^4 - 4a^2xy + x^4 = 0 \ldots (1)$ to find what values of x make y a maximum or minimum.

We have, differentiating

$$(y^3 - a^2x)\frac{dy}{dx} + x^3 - a^2y = 0,$$

$$\text{and } \therefore \frac{dy}{dx} = \frac{x^3 - a^2y}{a^2x - y^3}.$$

$y = \frac{x^3}{a^2}$ makes this zero, (at least if x be not zero, for then $\frac{dy}{dx} = \frac{0}{0}$, a case which we shall consider presently); now if $y = \frac{x^3}{a^2}$ we have in virtue of (1)

$$\frac{x^{12}}{a^8} - 4a^2\frac{x^4}{a^2} + x^4 = 0 \quad \text{or} \quad x^8 - 3a^8 = 0, \quad \text{dividing out } x^4;$$

$$\text{or } x = 3^{\frac{1}{8}}a, \quad \text{and } \therefore y = 3^{\frac{3}{8}}a;$$

these values put in the denominator of $\frac{dy}{dx}$ make it negative; therefore by (165) we may put

$$\frac{dy}{dx} = a^2y - x^3;$$

$$\therefore \frac{d^2y}{dx^2} = a^2\frac{dy}{dx} - 3x^3$$

$$= \text{negative, when we put } x = 3^{\frac{1}{8}}a \text{ and } \therefore \frac{dy}{dx} = 0;$$

$$\therefore x = 3^{\frac{1}{8}}a \text{ makes } y \text{ a maximum.}$$

Next, as to the value $x=0$ (and $\therefore y=0$) we find, putting $y=ux$ as in 158 and dividing out x^2, that $u_0=0$, and $\therefore \frac{dy}{dx}=0$

when $x = 0$, therefore $x = 0$ *may* give a maximum or minimum of y. To determine whether it does, put $y = ux^m$ as in 145, and we have

$$u^4 x^{4m} - 4a^2 u x^{m+1} + x^4 = 0;$$

$4m = m + 1$ gives $m = \frac{1}{3}$, and $4m$ or $m + 1 < 4$; $\therefore$ we may assume $m = \frac{1}{3}$, and this gives $u^4 - 4a^2 u = 0$ when $x = 0$; and $\therefore$ $u = (4a^2)^{\frac{1}{3}}$, we therefore have

$$y = (4a^2)^{\frac{1}{3}} x^{\frac{1}{3}} + R.$$

Again, $4m = 4$ makes $4m$ or $4 > m + 1$, and must therefore be rejected.

And again, $m + 1 = 4$ makes $m = 3$, and $\therefore$ $m + 1$ or $4 < 4m$; which gives us $u = \frac{1}{4a^2}$ when $x = 0$, and

$$\therefore\ y = \frac{x^3}{4a^2} + R.$$

Now by (168), both these expressions for y shew that $y = 0$ is neither a maximum nor a minimum value of y.

It appears from what has been just explained, that when we put $y = ux^m$ $\{$or $y - b = u\,(x - a)^m$ if $x = a$ and $y = b$ be the values we are concerned with$\}$, then we may immediately reject any value of m which is not an even number, or a fraction with an even numerator and odd denominator: bearing this in mind we may very readily obtain the maxima or minima values of y.

Lastly, $x = \frac{y^3}{a^2}$ makes $\frac{dy}{dx} = \infty$, and in virtue of (1) we have $y = 3^{\frac{1}{8}}a$, $x = 3^{\frac{3}{8}}a$.

Now assume $x = 3^{\frac{3}{8}}a + z$, and $y = 3^{\frac{1}{8}}a + uz^m$, and (1) becomes

$$(3^{\frac{1}{8}}a + uz^m)^4 - 4a^2\,(3^{\frac{1}{8}}a + uz^m)\,(3^{\frac{3}{8}}a + z) + (3^{\frac{3}{8}}a + z)^4 = 0,$$

which, bearing in mind the remark made in Appendix I respecting the rejection of certain terms, may be written

$$6 \,.\, 3^{\frac{2}{8}} a^2 u^2 z^{2m} - 4a^2 u z^{m+1} + 8 \,.\, 3^{\frac{1}{8}} a z = 0.$$

Here $2m = m + 1$ gives $m = 1$: $2m = 1$ gives $m = \frac{1}{2}$: and $m + 1 = 1$ gives $m = 0$. All these are to be rejected, since they are not even numbers or fractions with even numerators and odd denominators.

Hence it appears, that $x = 3^{\frac{1}{8}} a$ makes y a maximum, and neither $x = 0$ nor $x = 3^{\frac{3}{8}} a$ make y a maximum or minimum.

CHAPTER XII.

TANGENTS AND NORMALS TO CURVES. THE CURVATURE OF CURVES. THE EVOLUTE.

The Differential Calculus is of great use in various parts of analytical geometry. We have already seen in (26) that it puts us in possession of a general method of drawing tangents to curves. We now proceed to shew that it admits of, not only this, but many other important applications in analytical geometry.

Proposition. To find the equation to the tangent at any point of a curve.

173. If a right line SPQ (fig. 6) be drawn passing through any two points of a curve, and if SPT be its limiting position when Q approaches P, i. e. if the angle made by PT and PQ may be diminished *ad libitum* by sufficiently diminishing the arc QP; then PT is said to be a tangent to the curve at the point P. Let $x\,y$ be the co-ordinates of P, $x'y'$ these of Q, $\angle PTX = \psi$, then $\tan\psi$ is the limiting value of $\tan PRX$ when Q approaches P, i. e. of $\frac{y'-y}{x'-x}$ when x' approaches x, which, by definition of a differential coefficient, is $\frac{dy}{dx}$; hence

$$\tan\psi = \frac{dy}{dx}.$$

Hence if $x_{\prime}\,y_{\prime}$ be the co-ordinates of any point on the tangent PT, since it is a line passing through the point P and making an angle $\tan^{-1}\frac{dy}{dx}$ with the axis of x, we have for its equation

$$y_{\prime} - y = \frac{dy}{dx}(x_{\prime} - x).$$

Cor. The equation to the Normal.

174. The normal at the point P is the line PG (fig. 8) which passes through P at right angles to the tangent PT: its equation is therefore

$$y_{\prime} - y = -\frac{dx}{dy}(x_{\prime} - x).$$

Proposition. $ds^2 = dx^2 + dy^2$, s being the arc of the curve.

175. If we take s to represent the arc BP (fig. 6), s' the arc BQ, and c the cord PQ; then

$$\frac{s'-s}{x'-x} = \frac{s'-s}{c} \cdot \frac{c}{x'-x}$$

$$= \frac{s'-s}{c}\sqrt{1 + \left(\frac{y'-y}{x'-x}\right)^2}, \text{ since } c^2 = (x'-x)^2 + (y'-y)^2.$$

Now when x' approaches x the limiting value of $\frac{s'-s}{x'-x}$ is $\frac{ds}{dx}$, that of $\frac{s'-s}{c}$ is 1 by Lemma IX, and that of $\frac{y'-y}{x'-x}$ is $\frac{dy}{dx}$, hence by Lemma VIII, we have

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \text{ or } ds^2 = dx^2 + dy^2.$$

Cor. $dx = ds \cos\psi$, $dy = ds \sin\psi$.

176. We have $\cos PRX = \frac{x'-x}{c} = \frac{x'-x}{s'-s}\,\frac{s'-s}{c}$, and when x' approaches x the limiting value of PRX is ψ, that of $\frac{x'-x}{s'-s}$ is $\frac{dx}{ds}$, and that of $\frac{s'-s}{c}$ is 1; therefore

$$\cos\psi = \frac{dx}{ds}, \text{ or } dx = ds \cos\psi.$$

Similarly $\sin PRX = \frac{y'-y}{c} = \frac{y'-y}{s'-s} \cdot \frac{s'-s}{c}$;

$$\therefore \sin\psi = \frac{dy}{ds}; \quad \therefore dy = ds \sin\psi.$$

These formulæ may be very simply obtained by the method of Leibnitz.

177. If it be allowable to argue according to the method of Leibnitz, spoken of in (84), we may suppose Q taken so near P that the arc QP may be regarded as a right line coinciding with the tangent at P, and PO, OQ, PQ taken respectively equal to dx, dy, ds; these differentials being regarded as infinitely small: then we have immediately

$$dx = ds \cos\psi, \quad dy = ds \sin\psi;$$

$$\text{and therefore } dx^2 + dy^2 = ds^2, \text{ and } \frac{dy}{dx} = \tan\psi.$$

Which formulæ are all correct results in consequence of a compensation of errors such as that shewn to exist in (85).

Though this method of arriving at these formulæ is not sufficiently exact for the purposes of elementary demonstration, it is useful in enabling us to remember and make out these formulæ.

Subtangent, subnormal and normal.

178. The line MT (fig. 8) is called the subtangent, MG the subnormal, PG the normal.

Hence

$$\text{subtangent} = y\cot\psi = \frac{y\,dx}{dy},$$

$$\text{subnormal} = y\tan\psi = \frac{y\,dy}{dx},$$

$$\text{normal} = y\sec\psi = \frac{y\,ds}{dx}.$$

(For examples of the use of these formulæ see Appendix L.)

Proposition. To determine whether the concavity of a curve is turned upwards or

179. Since $\psi = \tan^{-1}\dfrac{dy}{dx}$, we have

$$\frac{d\psi}{dx} = \frac{1}{1 + \left(\dfrac{dy}{dx}\right)^2} \cdot \frac{d^2y}{dx^2}.$$

down-
wards, to
the right or
to the left.

Now $\dfrac{1}{1+\left(\dfrac{dy}{dx}\right)^2}$ is essentially a positive quantity;

hence $\dfrac{d\psi}{dx}$ has always the same sign as $\dfrac{d^2y}{dx^2}$.

Now if $\dfrac{d\psi}{dx}$ be positive, ψ increases when x increases, and therefore the concavity of the curve must be turned upwards, as in (fig. 9), but if $\dfrac{d\psi}{dx}$ be negative, ψ diminishes when x increases, and therefore the concavity of the curve is turned downwards, as in (fig. 10). Hence the concavity of the curve is turned upwards or downwards according as $\dfrac{d^2y}{dx^2}$ is positive or negative.

And in the same way it may be shewn that the concavity of the curve is turned to the right or to the left according as $\dfrac{d^2x}{dy^2}$ is positive or negative.

Proposi-
tion.
To deter-
mine points
of contrary
flexure.

180. Hence if at any point $\dfrac{d^2y}{dx^2}$ changes its sign, the concavity of the curve will change its direction, if the change of sign is from + to − the concavity which was before turned upwards will begin to be turned downwards, as at the point *P* (fig. 11); and if the change be from − to + the reverse will be the case, as at the point *P* (fig. 12).

A point where the concavity of the curve thus changes its direction is called a point of contrary flexure.

Hence, by Lemma XVII, $\dfrac{d^2y}{d^2x}$ must be 0 or ∞ at a point of contrary flexure; and therefore, to determine where such points are on the curve, we have only to find what values of x make $\dfrac{d^2y}{dx^2}$ 0 or ∞, and try whether x, in passing through each of these values makes $\frac{d^2y}{dx^2}$ change its sign or not; those for which there is a change of sign give points of contrary flexure, and those for which there is not are to be rejected.

It is evident that ψ is a maximum or minimum at a point of contrary flexure.

(For Examples, see Appendix M.)

Proposition. To determine the degree of curvature at any point of a curve. Index of curvature.

181. $\frac{d\psi}{dx}$ measures the rate at which the tangent changes its position as we go along the axis of x, and of course its magnitude depends in part upon what position the axis of x occupies with reference to the curve. $\frac{d\psi}{ds}$ measures the rate at which the tangent changes its position as we go along the curve, and its magnitude does not at all depend upon the position of the co-ordinate axes, but solely upon the nature of the curve: the greater therefore the curvature at any point, the greater will be $\frac{d\psi}{ds}$, and *vice versâ*. $\frac{d\psi}{ds}$ may therefore be called the *Index of curvature* at any proposed point, since it indicates the rate at which the tangent changes its position as we go along the curve; i.e. the degree of curvature of the curve at each point.

Proposition respecting the quantity ρ.

182. *If* PS, QS, *fig.* 13, *be two normals to a curve at the points* P *and* Q, *and if* ρ *be the limiting value of* PS *when* Q *approaches* P, *and* c *the chord joining* PQ; *then* ρ *is the limiting value of* $\frac{c}{\angle S}$ *when* Q *approaches* P.

$$\text{For} \quad \frac{c}{S} = \frac{c}{\sin S} \cdot \frac{\sin S}{S}$$

$$= \frac{PS}{\sin PQS} \cdot \frac{\sin S}{S}.$$

Now by sufficiently diminishing PQ we may evidently make $\angle PQS$ differ from $\frac{\pi}{2}$ *ad libitum;* therefore when Q approaches P, 1 is the limiting value of $\sin PQS$; 1 is also the limiting value of $\frac{\sin S}{S}$ by Lemma X, and we suppose ρ to be the limiting value of PS. Hence the limiting value of $\frac{c}{S}$ when Q approaches P is ρ. Q. E. D.

Cor.
Index of curvature is equal to $\frac{1}{\rho}$.

183. Let ψ and ψ' be the angles which the tangents at P and Q make with the axis of x, and s, s' the lengths of the arcs BP, BQ; then since $\psi' - \psi$ evidently $= \angle S$, we have

$$\frac{\psi' - \psi}{s' - s} = \frac{\angle S}{c} \cdot \frac{c}{s' - s}.$$

Now when Q approaches P, the limiting value of $\frac{\psi' - \psi}{s' - s}$ is $\frac{d\psi}{ds}$, that of $\frac{\angle S}{c}$ is $\frac{1}{\rho}$ by what has just been proved, and that of $\frac{c}{s' - s}$ is 1 by Lemma IX: hence, we have

$$\frac{d\psi}{ds} = \frac{1}{\rho};$$

i.e. *the index of curvature* $= \frac{1}{\rho}$.

ρ is called the radius of curvature for reasons we shall hereafter explain.

Proposition.
To find the value of ρ at any point of a given curve.

184. We may find ρ in terms of x and y as follows:

we have $\psi = \tan^{-1}\frac{dy}{dx}$; see (173).

$$\therefore\ d\psi = \frac{d\left(\frac{dy}{dx}\right)}{1 + \frac{dy}{dx^2}}$$

$$= \frac{dx^2}{ds^2}\, d\left(\frac{dy}{dx}\right), \text{ since } dx^2 + dy^2 = ds^2 \quad (175).$$

Hence, since $\frac{1}{\rho} = \frac{d\psi}{ds}$, we have

$$\frac{1}{\rho} = \frac{dx^2}{ds^3}\, d\left(\frac{dy}{dx}\right) = \frac{dx\,d^2y - d^2x\,dy}{ds^3} \text{......} (1).$$

If we suppose x the independant variable, this becomes

$$\frac{1}{\rho} = \frac{dx^2}{ds^3}\frac{d^2y}{dx} = \frac{dx\,d^2y}{ds^3} \text{.............} (2).$$

From the expressions (1) or (2) we may find $\frac{1}{\rho}$ when we know the curve.

Since $\frac{d\psi}{ds} = \frac{1}{\rho}$, it is clear from (179) that ρ is positive or negative according as the concavity of the curve at the point P is turned upward or downward.

(For examples, see Appendix N.)

Proposition. To determine the co-ordinates of the extremity of ρ.

185. Let P' (fig. 14) represent the limiting position of the point S in fig. 13, when Q approaches P; then PP' is the quantity we have denoted by ρ; let AL $(= \alpha)$, LP' $(= \beta)$, be the co-ordinates of P', PK parallel to LM. Then, since PP' evidently makes an angle $\frac{\pi}{2} - \psi$ with the line PK, we have

$$\left.\begin{aligned} \alpha &= x - \rho \sin \psi \\ \beta &= y + \rho \cos \psi \end{aligned}\right\} \text{.........} (1),$$

which equations determine the position of the point P' corresponding to the point xy of the curve.

Proposition. To find the curve traced out by P' when P moves along the curve BPQ.

186. It is evident that the point P' changes its position and traces out some curve when the point P is made to move along the curve BPQ; we may find the equation to the curve thus traced out as follows:

Since ρ, ψ, and y are all certain functions of x which we may find when the curve PQ is given, it is evident that we may eliminate x, y, ψ, ρ, from the equations (1), and so find a relation between α and β which is independent of x, and therefore holds for all positions of the point P. This relation is therefore the equation to the curve traced out by P'. This curve has some curious properties, as we proceed to shew.

Remarkable properties of this curve.

187. Differentiating the equations (1) we find

$$d\alpha = dx - \rho \cos\psi d\psi - d\rho \sin\psi,$$

$$d\beta = dy - \rho \sin\psi d\psi + d\rho \cos\psi.$$

$$\text{Now } \rho \cos\psi d\psi = \frac{ds}{d\psi}\frac{dx}{ds} d\psi \text{ (183, 176)},$$

$$= dx,$$

$$\text{and } \rho \sin\psi d\psi = dy \text{ similarly;}$$

$$\text{hence } d\alpha = -d\rho \sin\psi,$$

$$d\beta = \quad d\rho \cos\psi.$$

from which equations we find

$$\frac{d\beta}{d\alpha} = -\cot\psi \ldots\ldots (2)$$

$$d\alpha^2 + d\beta^2 = d\rho^2 \ldots\ldots\ldots\ldots (3).$$

First Property. The normal PP' touches the curve traced out by P' at the point P'.

188. Let $B'P'Q'$ (fig. 15) be the curve traced out by P'; then the equation to the tangent to this curve at the point P' is

$$y_1 - \beta = \frac{d\beta}{d\alpha}(x_1 - \alpha),$$

$$\text{or } y_1 - \beta = -\cot\psi . (x_1 - \alpha) \text{ by (2).}$$

Now this is evidently the equation to the line PP' since that line passes through the point P', and makes an angle $\psi + \frac{\pi}{2}$ with the axis of x. Hence the normal PP' touches the curve traced out by P' at the point P' as we have represented in the figure.

Second Property. The curve BPQ is traced out by the end of a string unwound off the curve $B'P'Q'$.

189. Again, if σ denote the length of the arc $B'P'$ of the curve $B'P'Q'$ measured from some fixed point B', we have

$$d\sigma = \sqrt{d\alpha^2 + d\beta^2} = \pm d\rho \text{ by (3):}$$

since ρ evidently decreases when σ increases in consequence of the way we have measured σ, we must take the lower of these signs, and therefore we have

$$d\sigma + d\rho = 0;$$

$$\text{and} \quad \therefore \ d(B'P' + P'P) = 0,$$

i.e. the length $B'P' + P'P =$ some constant, C suppose.

Now let one end of a string whose length is C be fastened at B', and the other end to the point of a pencil; and the string being full stretched let it be wound on the curve $B'P'$ till it touches it at P', and therefore coincides with $P'P$ in direction: then since the length of the string is C, which $= B'P' + P'P$, the pencil point will be just at P; and this is true whatever point of the curve P' may be. Hence if we unwind the string off the curve $B'P'$, the pencil point will trace out the curve BPQ.

$B'P'Q$ is on this account called the evolute of BPQ.

It is on this account that the curve $B'P'Q'$ is called the evolute of the curve BPQ.

(For examples see Appendix O.)

CHAPTER XIII.

POLAR FORMULÆ, DIFFERENTIALS OF AREAS, SURFACES, AND VOLUMES.

WE now proceed to investigate certain formulæ analogous to those of the last chapter, supposing the curve referred to polar co-ordinates.

Proposition. To find $\frac{ds}{d\theta}$.

190. Let APQ (fig. 16) be any curve referred to polar co-ordinates; $SP = r$, $PSA = \theta$; $SQ = r'$, $QSA = \theta'$, chord $PQ = c$, arc $AP = s$, arc $AP' = s'$: then

$$\frac{s' - s}{\theta' - \theta} = \frac{s' - s}{c} \, \frac{c}{\theta' - \theta}.$$

Now

$$c^2 = r'^2 + r - 2rr' \cos(\theta' - \theta)$$

$$= (r' - r)^2 + 4rr' \sin^2 z,$$

putting for brevity $z = \frac{\theta' - \theta}{2}$, and $\cos \theta' - \theta = 1 - 2 \sin^2 z$.

Hence $$\frac{s' - s}{\theta' - \theta} = \frac{s' - s}{c} \sqrt{\left(\frac{r' - r}{\theta' - \theta}\right)^2 + rr' \left(\frac{\sin z}{z}\right)^2}.$$

Let θ' approach θ, and therefore z zero; then the limiting value of $\frac{s' - s}{\theta' - \theta}$ is $\frac{ds}{d\theta}$, that of $\frac{s' - s}{c}$ is 1, that of $\frac{\sin z}{z}$ is also 1, and that of r' is r: therefore we have

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}, \quad \text{or } ds = \sqrt{dr^2 + r^2 d\theta^2}.$$

Cor. This result deduced from the rectangular formula.

191. This result we may also prove as follows.

By (175) $ds = \sqrt{dx^2 + dy^2}$,

$$\text{but} \quad x = r\cos\theta, \quad y = r\sin\theta;$$

$$\therefore\ dx = dr\cos\theta - r\sin\theta\, d\theta,$$

$$dy = dr\sin\theta + r\cos\theta\, d\theta;$$

$$\therefore\ dx^2 + dy^2 = dr^2 + r^2\, d\theta^2; \text{ and } \therefore\ ds = \sqrt{dr^2 + r^2\, d\theta^2}.$$

Proposition. To find p the perpendicular on the tangent from S.

192. Draw SY perpendicular to the line QPT which passes through the points P and Q, and draw PO perpendicular to SQ, then by similar triangles

$$SY = SQ\,.\,\frac{OP}{PQ} = r'\,\frac{r\sin(\theta' - \theta)}{c}$$

$$= r'r\,.\,\frac{\sin(\theta' - \theta)}{\theta' - \theta}\,.\,\frac{\theta' - \theta}{s' - s}\,.\,\frac{s' - s}{c}.$$

Now let θ' approach θ; then the limiting position of the line QPT will be the tangent at P; and therefore if p be the perpendicular from S upon that tangent, the limiting value of SY will be p. Hence, by Lemmas IX and X, we have

$$p = r^2\,\frac{d\theta}{ds}.$$

Cor. This result deduced from rectangular formulæ.

193. This result we may also prove as follows.

The equation to the tangent is

$$y_1 - y = \frac{dy}{dx}(x_1 - x),$$

and the perpendicular p upon this line from the origin is

$$\frac{x\dfrac{dy}{dx} - y}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}, \text{ or } \frac{x\,dy - y\,dx}{ds}, \text{ since } ds = \sqrt{dx^2 + dy^2};$$

hence, since $y = r\sin\theta$, $x = r\cos\theta$;

and therefore $dy = dr\sin\theta + r\cos\theta\, d\theta$,

$$dx = dr\cos\theta - r\cos\theta\, d\theta,$$

we have

$$xdy - ydx = r^2\, d\theta,$$

and therefore $p = \dfrac{r^2\, d\theta}{ds}$.

Proposition. To find the sine and cosine of the angle under the tangent and radius vector, viz. ϕ.

194. Sin $OQP = \dfrac{OP}{c} = \dfrac{r\sin(\theta' - \theta)}{c}$

$$= \frac{r\sin(\theta' - \theta)}{\theta' - \theta} \cdot \frac{\theta' - \theta}{s' - s} \cdot \frac{s' - s}{c}.$$

Now let θ' approach θ, then if ϕ be the angle which the tangent at P makes with the radius vector SP, the limiting value of OQP is ϕ. Hence we have

$$\sin\phi = \frac{rd\theta}{ds}.$$

Hence $\cos\phi = \sqrt{1 - \left(\dfrac{rd\theta}{ds}\right)^2}$

$$= \sqrt{\frac{ds^2 - r^2\, d\theta^2}{ds^2}}$$

$$= \frac{dr}{ds} \text{ by (191).}$$

Hence $\tan\phi = \dfrac{rd\theta}{dr}$.

Cor. p hence found.

195. Since $p = r\sin\phi$, we have from this

$$p = \frac{r^2 d\theta}{ds} \text{ as before.}$$

Proposition. To find $\tan\phi$ independantly of the preceding articles.

196. $$\tan OQP = \frac{OP}{OQ} = \frac{r\sin(\theta'-\theta)}{r' - r\cos(\theta'-\theta)}$$

Here put $\dfrac{\theta'-\theta}{2} = z$, and $\cos(\theta'-\theta) = 1 - 2\sin^2 z$,

$$\text{then } \tan OQP = \frac{r\sin(\theta'-\theta)}{r' - r + 2r\sin^2 z}$$

$$= \frac{\dfrac{r\sin(\theta'-\theta)}{\theta'-\theta}}{\dfrac{r'-r}{\theta'-\theta} + r\dfrac{\sin z}{z}\sin z}.$$

Now let θ' approach θ, and therefore z zero; then the limiting values of $\dfrac{\sin(\theta'-\theta)}{\theta'-\theta}$, $\dfrac{r'-r}{\theta'-\theta}$, $\dfrac{\sin z}{z}$, and $\sin z$, are respectively 1, $\dfrac{dr}{d\theta}$, 1, and zero; hence we have

$$\tan\phi = \frac{r}{\dfrac{dr}{d\theta}} = \frac{r\,d\theta}{dr}.$$

Cor. To find the polar subtangent.

197. The line drawn from S perpendicular to the radius vector to meet the tangent is called the polar subtangent; it evidently is equal to

$$r\tan\phi, \text{ or } \frac{r^2 d\theta}{dr}.$$

These formulæ proved by Leibnitz' method very simply.

198. If it be allowable to argue according to the method of Leibnitz, spoken of in (84), we may, as in (177), suppose

$$QP = ds, \quad QO = dr, \quad OP = r\,d\theta, \quad \angle OQP = \phi:$$

and hence we get

$$ds = \sqrt{r^2 d\theta^2 + dr^2}, \quad \sin\phi = \frac{r\,d\theta}{ds}, \quad \tan\phi = \frac{r\,d\theta}{dr},$$

$$p = r\sin\phi = \frac{r^2 d\theta}{ds}, \text{ polar subtangent} = r\tan\phi = \frac{r^2 d\theta}{dr}.$$

This method of arriving at these formulæ will enable us to make out any of them very readily, in case we forget them.

(For Examples of the use of these formulæ see Appendix P.)

Proposition. To find ρ in terms of r and p.

199. If ψ be the angle made by PT with SA, then we have

$$\psi = \theta + \phi, \text{ and } \therefore d\psi = d\theta + d\phi.$$

Now $p = r \sin \phi$, and

$$\therefore dp = dr \sin \phi + r \cos \phi \, d\phi$$

$$= dr \frac{r d\theta}{ds} + \frac{r dr}{ds} d\phi$$

$$= \frac{r dr}{ds} (d\theta + d\phi) = \frac{r dr}{ds} d\psi$$

$$= \frac{r dr}{\rho}, \text{ since } \frac{d\psi}{ds} = \frac{1}{\rho} \quad (183);$$

$$\text{hence } \rho = \frac{r dr}{dp}.$$

Proposition. To find ρ in terms of r and θ.

200. We have as in (199)

$$\psi = \phi + \theta = \tan^{-1} \frac{r d\theta}{dr} + \theta,$$

hence differentiating, supposing $d\theta$ constant, we have

$$d\psi = d\theta \left(\frac{1 - \dfrac{r d^2 r}{dr^2}}{1 + \dfrac{r^2 d\theta^2}{dr^2}} + 1 \right)$$

$$= d\theta \frac{2dr^2 - r d^2 r + r^2 d\theta^2}{ds^2}, \text{ since } ds^2 = dr^2 + r^2 d\theta^2,$$

$$\text{hence } \rho = \frac{ds}{d\psi} = \frac{ds^3}{d\theta\,(2dr^2 - r d^2r + r^2 d\theta^2)}$$

$$\text{or } \frac{ds^3}{d\theta\,(dr^2 + ds^2 - r d^2 r)}.$$

Cor. The same result deduced from rectangular formulæ.

201. This result may also be arrived at thus.

We have by (186) $\rho = \dfrac{ds^3}{d^2y\,dx - d^2x\,dy}$.

$$\text{Now}\quad dy = dr\sin\theta + r\cos\theta\, d\theta,$$

$$d^2y = d^2r\sin\theta + 2dr\cos\theta\, d\theta - r\sin\theta\, d\theta^2,$$

$$dx = dr\cos\theta - r\sin\theta\, d\theta.$$

Hence

$$d^2y\,dx = (dr d^2r - 3r dr d\theta^2)\sin\theta\cos\theta$$
$$+ 2\,dr^2\cos^2\theta\,.\,d\theta - r\,(d^2r - r d\theta^2)\sin^2\theta\,.\,d\theta.$$

Now $d^2x\,dy$ is evidently got from this by putting $\frac{\pi}{2} - \theta$ instead of θ; hence we evidently have, putting $\sin\theta$ for $\cos\theta$, $\cos\theta$ for $\sin\theta$, $-d\theta$ for $d\theta$, and subtracting

$$d^2y\,dx - d^2x\,dy = d\theta\,(2\,dr^2 - r d^2r + r^2 d\theta^2),$$

$$\text{and } \therefore\ \rho = \frac{ds^3}{d\theta\,(2dr^2 - r d^2r + r^2 d\theta^2)}.$$

Proposition. To find whether the concavity of the curve is turned to or from the pole.

202. If p increases when r increases, it is easy to see that the concavity of the curve is turned towards the pole, and if p diminishes when r increases, it is turned from the pole. Hence, by Lemma XVIII., the concavity of the curve is turned towards or from the pole according as $\dfrac{dp}{dr}$ is positive or negative.

(For Examples see Appendix P.)

The use of the following formulæ respecting the differentials of areas, surfaces, and volumes cannot be shewn without the assistance of the Integral Calculus.

The Integral Calculus is the inverse of the Differential Calculus, its object being to find the quantity from which a given differential is derived. When the student becomes acquainted with it, he will perceive the importance of these formulæ.

Proposition. To find the differential of an area.

203. Let BPQ (fig. 17) be any curve, $AM=x$, $MP=y$, $AN=x'$, $NQ=y'$, the co-ordinates of any two points P and Q of it. Let A denote the area BPM and A' the area BQN; draw PO and QR parallel to MN to meet NQ and MP in O and R.

Then area $MPQN$ lies between $MRQN$ and $MPON$,

i.e. $A'-A$ lies between $y'(x'-x)$ and $y(x'-x)$;

$$\therefore \frac{A'-A}{x'-x} \text{ lies between } y' \text{ and } y.$$

Now $y'=y$ when $x'=x$, hence by Lemma VII. y is the limiting value of $\dfrac{A'-A}{x'-x}$ when x' approaches x, i.e.

$$\frac{dA}{dx}=y, \quad dA=y\,dx.$$

Proposition. To find the differential of a solid of revolution.

204. Let V and V' be the volumes generated by the revolution of the areas BPM and BQN (fig. 17) round AN, then

vol. gen. by $MPQN$ lies between vol. gen. by $MPON$ and vol. gen. by $MRQN$,

i.e. $V'-V$ lies between $\pi y^2(x'-x)$ and $\pi y'^2(x'-x)$;

$$\therefore \frac{V'-V}{x'-x} \text{ lies between } \pi y^2 \text{ and } \pi y'^2;$$

$$\therefore \text{ as before } \frac{dV}{dx}=\pi y^2, \quad dV=\pi y^2\,dx.$$

Proposition. To find the differential

205. Let S and S' denote the surfaces generated by the revolution of the arcs BP and BQ about AN (fig. 17); let s denote the former arc and s' the latter; produce PO

of a surface of revolution.

to U and QR to T, so that PT and QU shall each be equal to the arc PQ. Then we may evidently assume that

surf. gen. by PQ lies between surf. gen. by PT and surf. gen. by QU,

i.e. $S' - S$ lies between $2\pi y(s' - s)$ and $2\pi y'(s' - s)$,

$$\therefore \frac{S' - S}{s' - s} \text{ lies between } 2\pi y \text{ and } 2\pi y'.$$

$$\text{Hence as before } \frac{dS}{ds} = 2\pi y, \text{ or } dS = 2\pi y\sqrt{dx^2 + dy^2}.$$

Proposition. To find the differential of a polar area.

206. Let APQ (fig. 18) be a curve referred to polar co-ordinates, $SP = r$, $SQ = r'$, $PSA = \theta$, $QSA = \theta'$: let A and A' be the areas ASP and ASQ; describe the circular arcs PO and QR round S as center. Then

area SPQ lies between area SPO and area SQR,

$$\text{i.e. } A' - A \text{ lies between } \frac{r^2(\theta' - \theta)}{2} \text{ and } \frac{r'^2(\theta' - \theta)}{2};$$

$$\therefore \frac{A' - A}{\theta' - \theta} \text{ lies between } \frac{r^2}{2} \text{ and } \frac{r'^2}{2}.$$

$$\text{Hence as before } \frac{dA}{d\theta} = \frac{r^2}{2}.$$

Cor. The same result deduced from the rectangular formula.

207. This result may also be obtained as follows: draw PM perpendicular to SM, then $SM = x$ and $PM = y$ will be the rectangular co-ordinates of P, let B be the area APM, then by 203

$$dB = y\,dx,$$

$$\text{but } B = \text{area } SPM - \text{area } SPA$$

$$= \frac{xy}{2} - A;$$

$$\therefore \frac{x\,dy + y\,dx}{2} - dA = y\,dx;$$

$$\therefore dA = \frac{x\,dy - y\,dx}{2}.$$

Hence as in (193),

$$dA = \frac{r^2 d\theta}{2}.$$

$2dA = xdy - ydx.$

208. The result

$$2dA = xdy - ydx$$

is of importance.

CHAPTER XIV.

ASYMPTOTES.

An Asymptote, what.

209. It often happens, in the case of curves which have infinite branches, that when the point of contact is moved off to an infinite distance from the origin, the tangent remains at a finite distance, or, to speak more accurately, approaches a certain limiting position which is at a finite distance from the origin. In such a case the tangent or rather its limiting position is called an asymptote.

Proposition. To determine the equation to an asymptote of a curve.

210. The equation to an asymptote, when one exists, may be thus determined.

In the equation to the tangent, which may be written in the form

$$y_{,} = \frac{dy}{dx} x_{,} + y - \frac{dy}{dx} x,$$

$$\text{put } \frac{y}{x} = u, \quad \text{and therefore } \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{du}{dx},$$

and it evidently becomes

$$y_{,} = \left(\frac{y}{x} + x\frac{du}{dx}\right) x_{,} - x^2 \frac{du}{dx},$$

put moreover

$$x = \frac{1}{z}, \quad \text{and } \therefore \ dx = -\frac{dz}{z^2} = -x^2 dz,$$

and the equation reduces to the following form

$$y_{,} = \left(u - z\frac{du}{dz}\right) x_{,} + \frac{du}{dz}.$$

Now if the curve be of such a nature that when x approaches ∞, i. e. when z approaches zero, the limiting values of u and $\dfrac{du}{dz}$ are finite quantities A and B; then the limiting form which the equation to the tangent assumes is

$$y_{,} = Ax_{,} + B,$$

which therefore is the equation to the asymptote.

How to find A and B.

211. It is very easy to find A and B in all cases from the equation to the curve, by putting $x = \dfrac{1}{z}$, and $y = \dfrac{u}{z}$ in that equation, and so finding a relation between u and z: then A being the limiting value of u when z approaches zero may be found by putting $z = 0$, and B, which is the limiting value of $\dfrac{du}{dz}$ when z approaches zero, may be found by differentiating the equation and putting $z = 0$.

Example.

Let $x^3 + 3axy - y^3 = 0$ be the equation to the curve; then putting $x = \dfrac{1}{z}$, $y = \dfrac{u}{z}$ and multiplying by z^3 we find

$$1 + 3auz - u^3 = 0,$$

$$\text{and differentiating } 3au + (3az - 3u^2)\frac{du}{dz} = 0;$$

in these equations put $z=0$, and $\therefore u=A$, $\dfrac{du}{dz} = B$, and we find

$$1 - A^3 = 0, \quad 3aA - 3A^2B = 0;$$

$$\therefore \; A = 1, \quad B = \cancel{1}, \; a$$

and the equation to the asymptote is therefore

$$y = x + a.$$

There may be an asymptote corresponding to a finite value of x which must be found otherwise.

212. If A and B turn out to be impossible or infinite, then there is no asymptote corresponding to an infinite value of the abscissa: but there may be one corresponding to a finite value of x, for the point of contact may go off to an infinite distance from the origin for a finite value of x, y of course becoming infinite. Such an asymptote cannot be found by the

method just explained, since in that method we suppose x infinite; but it is easy to see that whenever the ordinate becomes infinite for a finite value of x, it becomes an asymptote, and therefore if a be a finite value of x which makes y infinite, the line whose equation is $x = a$ is an asymptote. Therefore all we have to do in a case of this kind is to put $y = \frac{1}{z}$ in the equation to the curve, and then make $z = 0$; and if this gives us a finite value of x, a suppose, $x = a$ is the equation to the asymptote.

Example. Let the equation to the curve be

$$y^2 x - x^3 - 2ay^2 = 0,$$

putting $y = \frac{u}{z}$, $x = \frac{1}{z}$ and multiplying by z^3 we find

$$u^2 - 1 - 2au^2 z = 0,$$

$$\text{and differentiating } (2u - 4auz)\frac{du}{dz} - 2au^2 = 0,$$

in these equations putting $z = 0$, and $\therefore u = A$, $\frac{du}{dx} = B$, we find

$$A^2 - 1 = 0, \quad B - aA = 0,$$

$$\therefore \; A = \pm 1, \quad B = \pm a,$$

hence there are two asymptotes corresponding to $x = \infty$ whose equations are

$$y = x + a, \quad \text{and} \quad y = -x - a.$$

Moreover, putting $y = \frac{1}{z}$ in the original equation and multiplying by z^2, we find

$$x - x^3 z^2 - 2a = 0,$$

and therefore when $z = 0$, $x = 2a$, which is a finite value; hence there is another asymptote whose equation is

$$x = 2a.$$

When therefore our object is to find all the asymptotes of a curve, we must not forget to try whether there is one corresponding to a finite value of x.

213. We may consider asymptotes somewhat differently, and perhaps more simply, in the following manner.

Asymptotes considered somewhat differently.

By putting $\frac{1}{z}$ for x, and $\frac{u}{z}$ for y in the equation to the curve, we find a relation between u and z; and if u and $\frac{du}{dz}$ have finite values A and B when $z = 0$, we may by Taylor's Theorem assume that

$$u = A + Bz + R;$$

where R is some quantity, such that zero is the limiting value of $\frac{R}{z}$ (i. e. of Rx) when z approaches zero, i. e. when x approaches infinity.

Therefore, restoring x and y, we may assume that

$$y = Ax + B + Rx.$$

Now let BP, fig. 19, represent the curve, $AM = x$, $MP = y$: also let $B'P'$ be the line whose equation is

$$y' = Ax + B, \text{ where } y' = MP',$$

$$\text{then } PP' = y - y' = Rx;$$

therefore, since zero is the limiting value of Rx when x approaches ∞, PP' may be diminished *ad libitum* by sufficiently increasing x. Therefore the line $B'P'$ continually approaches the curve BP as we go off to an infinite distance from the origin, but never actually meets it, though we may make PP' as small as we please. A line thus circumstanced is called an Asymptote, and we may find its equation just as before, since A and B are the same quantities as before*.

214. We may determine the asymptotes of a polar curve as follows:

Proposition.

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* See note, page 154.

To determine the asymptotes to a polar curve by means of what has been proved.

Let the equation to the curve be put in the form

$$r = \frac{1}{f(\theta)} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

then, since $x = r\cos\theta$, $y = r\sin\theta$, we have, putting $\frac{1}{z}$ for x, and $\frac{u}{z}$ for y,

$$z = \frac{1}{r\cos\theta} = \frac{f(\theta)}{\cos\theta} \ldots (1) \quad u = \tan\theta \ldots (2);$$

$$\text{and } \therefore \frac{du}{dz} = \frac{d\tan\theta}{d\left\{\dfrac{f(\theta)}{\cos\theta}\right\}} = \frac{1}{f'(\theta)\cos\theta + f(\theta)\sin\theta} \ldots (3).$$

Now let α be a value of θ which makes r infinite, and therefore

$$f(\theta)\left(\text{which } = \frac{1}{r}\right) = 0;$$

then $\theta = \alpha$ makes $z = 0$ by (1)*; and therefore if A and B be the values of u and $\frac{du}{dz}$ when $z = 0$, we have by (2) and (3),

$$A = \tan\alpha, \quad B = \frac{1}{f'(\alpha)\cos\alpha};$$

and therefore the rectangular equation to the asymptote required is

$$y = \tan\alpha \,.\, x + \frac{1}{f'(\alpha)\cos\alpha}.$$

$$\text{or} \quad x = -\frac{1}{f'\left(\frac{\pi}{2}\right)} \text{ if } \alpha = \frac{\pi}{2}; \text{ see note.}$$

* Except when $\alpha = \frac{\pi}{2}$, in which case z assumes the form $\frac{0}{0}$ when $\theta = \alpha$, and its limiting value when θ approaches α is $-f\left(\frac{\pi}{2}\right)$; $\therefore x = -\frac{1}{f\left(\frac{\pi}{2}\right)}$ makes r and therefore y infinite; therefore by 212 $x = -\frac{1}{f'\left(\frac{\pi}{2}\right)}$ is the equation to the asymptote.

215. We may also obtain this result as follows. Cor. Same result obtained independantly.

Let a be the angle which the asymptote makes with the prime radius; then it is evident that $\theta = a$ ought to make r infinite, and therefore $f(a)$ must be zero. Let the rectangular equation to the asymptote be

$$y = \tan a\,(x + c) \;\ldots\ldots\ldots\ldots\; (2),$$

which, putting $r'\cos\theta$ $r'\sin\theta$ for x and y, becomes

$$r' = \frac{c \sin a}{\sin(\theta - a)};$$

$$\text{hence } \frac{r'}{r} = c \sin a \cdot \frac{f(\theta)}{\sin(\theta - a)} = \frac{0}{0} \text{ when } \theta = a.$$

Now the limiting value of $\frac{r'}{r}$ when θ approaches a ought to be unity, from the nature of an asymptote; but by the usual method of vanishing fractions this limiting value is $c \sin a f'(a)$.

Hence we have $c \sin a f'(a) = 1$;

$$\text{and } \therefore\; c \sin a = \frac{1}{f'(a)}.$$

The equation to the asymptote is therefore $\{$substituting for c in $(2)\}$,

$$y = x \tan a + \frac{1}{\cos a f'(a)}.$$

We have here obtained the rectangular equation to the asymptote, because it is easier to make use of it than the polar.

Let $r = \frac{a}{\theta}$ be the equation to the curve; here Example 1.

$$f(\theta) = \frac{\theta}{a}; \text{ and } \therefore\; a = 0; \text{ also } f'(\theta) - \frac{1}{a}; \;\therefore\; f'(a) = \frac{1}{a};$$

therefore the equation to the asymptote is $y = a$, which represents a line parallel to the axis of x at a distance a above it. See fig. 20.

Let $r = \frac{a(e^2 - 1)}{1 - e \cos\theta}$ (a hyperbola referred to focus), Example 2.

$$\text{here}\quad f(\theta) = \frac{1 - e\cos\theta}{a(e^2 - 1)};$$

$$\text{and} \therefore \cos\alpha = \frac{1}{e},\quad \sin\alpha = \pm\sqrt{1 - \frac{1}{e^2}} = \pm\frac{\sqrt{e^2 - 1}}{e},$$

$$\tan\alpha = \pm\sqrt{e^2 - 1}.$$

$$\text{Also}\quad f'(\theta) = \frac{e\sin\theta}{a(e^2 - 1)};\quad \therefore f'(\alpha) = \pm\frac{1}{a\sqrt{e^2 - 1}};$$

hence the equation to the asymptote is

$$y = \pm\sqrt{e^2 - 1}\,.\,x \pm ae\sqrt{e^2 - 1},$$

$$\text{or}\quad y = \pm\frac{b}{a}(x + ae),$$

which shews that there are two asymptotes making angles $\tan^{-1}\frac{b}{a}$, and $-\tan^{-1}\frac{b}{a}$ with the axis of x, and meeting it at a distance ae behind the origin.

Asymptotic circle, what.

216. It sometimes happens that r assumes a finite value, c suppose, when we put $\theta = \infty$ in the polar equation to a curve: it is easy to see that in such a case if we describe a circle round the pole with radius c, we may by continually increasing θ make the curve approach as near as we please to this circle without ever actually meeting it. Such a circle is called an asymptotic circle.

Example.

Let $r = \frac{a\theta}{\pi + \theta}$; then $\theta = \infty$ makes $r = a$; therefore by continually increasing θ we may diminish $r \sim a$ *ad libitum*, and therefore make the curve approach as near to the circle as we please without ever actually meeting it, since it requires an infinite number of revolutions of r to make $r = a$.

The circle in this case is an exterior asymptote, since r is evidently always less than a, (see fig. 21).

If $r = \frac{a\theta}{\theta - \pi}$ be the equation, r is always greater than a (at least when θ is taken large enough), and therefore the circle is an exterior asymptote.

CHAPTER XV.

ON THE METHOD OF TRACING THE GENERAL FORM OF A CURVE FROM ITS EQUATIONS.

It is often necessary to make out and trace the general form of a curve from its equation, without actually calculating its exact dimensions, or ascertaining the precise positions of its remarkable points: we now proceed to state how this may be done in most cases.

How to trace a curve when we can find y in terms of x.

217. If we can find y in terms of x, it is easy in general to trace the general form of the curve by determining the values of x which make $y=0$ or ∞, the corresponding values of $\frac{dy}{dx}$, and the signs which y has between its zero and infinite values. We may do this in the following manner, viz. In one column write in order the values of x which make $y=0$ or ∞, and in addition to these, the value $x=\infty$: opposite to these values, in another column, write down the corresponding values of y, and between each two put the sign which y has between them (y will always have the same sign between each two of these values, since it can only change its sign in passing through 0 or ∞): and in a third column put the corresponding values of $\frac{dy}{dx}$. By means of such a table it will be easy, in most cases, to trace the general form of the curve, as the following example will shew.

Example.

218. To trace the general form of the curve whose equation is

$$y=\frac{x^2}{a}\frac{x+a}{x-a};$$

here y is zero when $x=0$ or $-a$, and infinite when $x=a$: hence the values of x to be written down are $-a$, 0, a, ∞;

the corresponding values of y are 0, 0, ∞, ∞: and by differentiating, or rather by the method given in the note*, we shall find that the corresponding values of $\frac{dy}{dx}$ are $-\frac{1}{2}$, 0, ∞, ∞†; also when x is $< -a$ the sign of y is +, when

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How to find $f'(a)$ very readily when $f(a) = 0$ or ∞.

* To find the value of the differential coefficient of $f(x)$ when $x = a$, supposing a to be a value of x which makes $f(x) = 0$ or ∞, we have only to divide $f(x)$ by $x - a$ and then put $x = a$: for if $f(a) = 0$, the limiting value of $\frac{f(x)}{x-a}$ when x approaches a is $f'(a)$ by Lemma XIX.; and if $f(a) = \infty$, $\frac{f(x)}{x-a}$ becomes ∞ when $x = a$, and thus gives the proper value of $f'(a)$, which by Lemma XXIII. we know to be ∞. *In either case therefore, if we divide* f(x) *by* x − a *and put* x = a, *the result will be the proper value of* f′(a).

Thus let $f(x) = \frac{x^2}{a}\frac{x+a}{x-a}$; then if we divide by $x + a$ and put $x = -a$ we obtain immediately $f'(-a) = -\frac{1}{2}$. If we divide by x and put $x = 0$, we find $f'(0) = 0$. If we divide by $x - a$ and put $x = a$, we find $f'(a) = \infty$.

To find y and $\frac{dy}{dx}$ when $x = \infty$.

† The values of y and $\frac{dy}{dx}$ when $x = \infty$ may be easily found thus.

Suppose y to be in the form of a fraction $\frac{u}{v}$, put u in the form $x^m(A + R)$ and v in the form $x^n(B + R')$, where R and R' are quantities which vanish when $x = \infty$;

$$\text{then}\quad y = x^{m-n}\frac{A+R}{B+R'};$$

and therefore, if $m = n$, $y = \frac{A}{B}$ when $x = \infty$; if m be $< n$, $y = 0$ when $x = \infty$; and if m be $> n$, $y = \infty$ when $x = \infty$. It is always quite easy to put an algebraical function of x in the form $x^m(A + R)$ by examining which term contains the highest power of x, and making that power appear as a factor of the whole. Thus

$$x^2 + 3x^3 + 6 = x^3\left(3 + \frac{1}{x} + \frac{6}{x^3}\right) = x^3(3 + R),$$

$$a^2x + ax\sqrt{x^2 - a^2} + ax^2 = x^2\left(a + a\sqrt{1 - \frac{a^2}{x^2}} + \frac{a^2}{x}\right) = x^2(2a + R).$$

To find $\frac{dy}{dx}$ when $x = \infty$; it is easy to see that if $m - n$ be > 1, supposing $y = x^{m-n}\frac{A+R}{B+R'}$, then $\frac{dy}{dx} = \infty$ when $x = \infty$; if $m - n = 1$, then $\frac{dy}{dx} = \frac{A}{B}$ when $x = \infty$; and if $m - n$ be < 1, then $\frac{dy}{dx} = 0$ when $x = \infty$. So then we may very readily find $\frac{dy}{dx}$ at the same time that we find y corresponding to the value $x = \infty$.

For example, if $y = \frac{x^2}{a}\frac{x+a}{x-a}$ we may immediately put it in the form

x is $> -a$ and < 0 it is $-$, when x is > 0 and $< a$ it is $-$, when x is $> a$ and $< \infty$ it is $+$. Hence the table will be as follows.

x	y	$\frac{dy}{dx}$
	$+$	
$-a$	0	$-\frac{1}{2}$
	$-$	
0	0	0
	$-$	
a	∞	∞
	$+$	
∞	∞	∞

$$y = \frac{x^3\left(1+\frac{a}{x}\right)}{x\left(a-\frac{a^2}{x}\right)} = x^2\,\frac{1+R}{a+R'};$$

$$\therefore\ y = \infty,\ \text{and}\ \frac{dy}{dx} = \infty,\ \text{when}\ x = \infty.$$

For a second example, let $y = x\left(\frac{x+a}{x-a}\right)^2$;

$$\text{then}\quad y = x\left(\frac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^2 = x\,\frac{1+R}{1+R'};$$

$$\therefore\ y = \infty,\ \text{and}\ \frac{dy}{dx} = 1,\ \text{when}\ x = \infty;$$

For a third example, let $y = \frac{a^2\sqrt{x^2-a^2}}{(x-b)(x-c)}$;

$$\text{then}\quad y = \frac{xa^2\sqrt{1-\frac{a^2}{x^2}}}{x^2\left(1-\frac{b}{x}\right)\left(1-\frac{c}{x}\right)} = \frac{1}{x}\cdot\frac{a^2+R}{1+R'};$$

$$\therefore\ y = 0,\ \text{and}\ \frac{dy}{dx} = 0,\ \text{when}\ x = \infty.$$

From this table it is immediately evident that fig. 22 represents the curve. For if we take AX, AY as axes, $AB = a$, $AC = -a$; then the table shews that at the point C the curve passes from the positive to the negative side of the axis of x at an angle $\tan^{-1}(-\frac{1}{2})$, as is represented in the figure; that it touches the axis of x at A, but does not cross it; that it goes off to an infinite distance when $x = AB$; and when x becomes greater than AB it appears on the positive side of the axis of x, evidently in the form FPG, for y is infinite when $x = AB$, becomes and continues positive when x is greater than AB, and becomes infinite again when x is infinite; y therefore must decrease as x gets greater than AB as far as a certain point after which it must increase again in order that it may go off to infinity when x becomes infinite without becoming negative; from which it is evident that the curve must be in the form FPG. Since y and $\frac{dy}{dx}$ are both infinite when $x = \infty$, they must be very large when x is either a very large positive or a very large negative quantity, and therefore the curve must run off to infinity on the positive and negative side of the axis of y in the manner represented in the figure.

Representation by figures of different cases which may occur.

219. In the plates marked M the Student will find some of the different cases which may occur in tracing a curve represented by figures, to which he will find it useful to refer. Under each figure is set down the corresponding line in the table from which the figure is deduced. By means of this plate the use of the table will appear evident.

To determine the form of the curve more precisely, we must examine whether it has any points of contrary flexure.

220. Thus we are able to trace the *general* form of the curve; but this method does not always shew us the points of contrary flexure of the curve; to determine which we must resort to the method given in (180). Thus in the present case, differentiating y twice, we get

$$\frac{d^2y}{dx^2} = 2\frac{(x-a)^3 + 2a^3}{(x-a)^3},$$

the only real values of x which make this $= 0$ or ∞ are $x = a$, $x = a - 2^{\frac{1}{3}}a$ or $-a(2^{\frac{1}{3}} - 1)$: so that there can be only two points of contrary flexure; which indeed the figure shews, for there must evidently be a change of flexure somewhere between C and A, since the concavity of the curve is turned upward at C and downward at A; also there is evidently a change of flexure at B. There is no change of flexure elsewhere, and therefore the figure represents the precise form of the curve.

We may find the maximum and minimum ordinates, in order to determine the form of the curve more minutely.

221. In order to determine still more minutely the form of the curve, we may find the points where the ordinate is a maximum or a minimum; to do this in the present case, we have

$$\frac{dy}{dx} = 2\frac{x^3 - ax^2 - a^2x}{a(x-a)^2}.$$

and therefore, when $\frac{dy}{dx} = 0$, we have

$$x^3 - ax^2 - a^2x = 0,$$

which gives $x = 0$, and $x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + a^2}$, or $\frac{a}{2}(1 \pm \sqrt{5})$; which three values correspond to the points A, Q, and P.

Asymptotes are useful in giving us an idea of the manner in which the branches of a curve go off to infinity. Example 2.

222. When the curve has asymptotes, it is often useful to find them; for they enable us to see more clearly the nature of the infinite branches of the curve; for example, take the equation

$$y = x\frac{(x+a)^2}{(x-a)^2}.$$

Forming a table as in the former case, we find

x	y	$\frac{dy}{dx}$
	$-$	
$-a$	0	0
	$-$	
0	0	1
	$+$	
a	∞	∞
	$+$	
∞	∞	1

Since* $y = x\left(\dfrac{1+\frac{a}{x}}{1-\frac{a}{x}}\right)^2 = x\left(1 + \dfrac{4a}{x} + \&c.\right) = x + 4a + R,$

where R becomes 0 when $x = \infty$, the curve has an asymptote whose equation is

$$y = x + 4a.$$

Hence, take AC (fig. 23) $= a$, $AB = a$, $AN = 4a$, and draw the line $DMNG$, making $\angle 45^0$ with AX, and the line BF perpendicular to AX. Then $DMNG$ is an asymptote to the curve; the curve comes from infinity below the axis of x, touches it at C, but does not cross it, crosses it at A at an angle 45^0, goes off to infinity as it approaches the line BF, appears on the positive side of AX when it passes BF, and then turns off towards the line $DMNG$ as its asymptote.

(For more Examples, see Appendix S.)

How to trace a curve from its polar equation.

223. We may trace the general form of a curve referred to polar co-ordinates in a similar manner, by putting down the values of θ which make r zero or infinity; but as there are often no such values, or only very few of them, a table of those values is not in general sufficient to indicate the form of the curve. We may in most cases perceive the form of the curve by considering whether r increases or diminishes as it turns round; and this we may see, either by simple inspection, or by finding $\dfrac{dr}{d\theta}$ and examining whether its sign is + or −, which will tell us whether r increases or diminishes with θ.

There is no use however in putting down the values of $\dfrac{dr}{d\theta}$ as we did those of $\dfrac{dy}{dx}$ in the case of rectangular curves: but it will sometimes be advisable to put down the signs which

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* It very often happens, as in the present example, that we can immediately put y in the form $Ax + B + R$, where R becomes 0 when $x = \infty$; in such a case we have no need to resort to the general method of finding asymptotes given in the preceding Chapter.

$\frac{dr}{d\theta}$ has, in order to see whether r is increasing or diminishing with θ. When r is negative, $\frac{dr}{d\theta} = +$, indicates a diminution in the absolute numerical value of r, and *vice versâ*.

It is sometimes necessary, in order to make out properly the nature of the curve, to put down a few of the values of r corresponding to such values of θ, as $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ ... $-\frac{\pi}{2}$, $-\frac{3\pi}{2}$, ... &c.

When $\theta = a$ makes $r = 0$, it is clear that there is a tangent at the origin making an angle a with the prime radius. When $\theta = a$ makes $r = \infty$, we must find the corresponding asymptote.

It is usual to neglect the negative values of r in tracing polar curves, but there is no reason whatever to justify this omission, and we shall therefore always consider these values to be, as they really are, just as important as the positive values.

Positive and negative values of r, how measured.

224. In measuring positive and negative values along r, it will save mistakes, to suppose an arrow-head fixed upon the radius vector to revolve with it as θ increases; then this arrow-head, supposing it to point in the positive direction when $\theta=0$, will always continue to point in the positive direction, whatever be the value of θ. Thus, in fig. 24, let $\theta = \frac{\pi}{4}$, and then from S to P is the positive direction, and from S to P' the negative: in fig. 25, let $\theta = \pi + \frac{\pi}{4}$, and then from S to P is the positive direction: in fig. 26, let $\theta = \frac{3\pi}{2} + \frac{\pi}{4}$, and then from S to P is the positive direction: and in fig. 27, let $\theta = -\frac{\pi}{4}$, and then from S to P is the positive direction.

Thus we see that the positive direction with reference to a revolving line is not a fixed direction in space, but depends on the angle at which the radius vector is inclined to its original position. The principle on which what we have just stated depends belongs to a different part of elementary mathematics, and therefore we assume it here without explanation.

Representation by figures of different cases.

Bearing these considerations in mind, it will in general be easy to trace the form of a curve from its polar equation, as will appear by the following examples. The Student will find in the plates marked M a representation by figures of some of the different cases which may occur, and the corresponding line of the table under each: to which he will find it useful to refer.

Example 1.

225. Let $r = a\theta$ be the equation to the curve.

Here then is no use in forming a table; we see immediately that as θ increases, r continually increases; when $\theta = 0$, $r = 0$; and when θ is negative, r is negative.

Hence the curve passes through the pole and touches the prime radius when $\theta = 0$, and then continually recedes from the pole as r turns round. When θ = a negative angle, ASP suppose (fig. 28), then r is negative, and therefore must be measured in the direction SP'. Hence the curve is evidently of the form represented in the figure, the dotted part corresponding to the negative values of θ being similar to that corresponding the positive.

Example 2.

226. Let $r = a\dfrac{\theta^2 - \pi^2}{\theta}$.

Here $\dfrac{dr}{d\theta} = a\left(1 + \dfrac{\pi^2}{\theta^2}\right)$ which is essentially positive: hence r always increases with θ.

When $\theta = 0$, $r = \infty$, and by the method explained in the last chapter $y = a\pi^2$ is the equation to the corresponding asymptote.

Forming a table, we have

θ	r	$\frac{dr}{d\theta}$	
	$-$	$+$	
$-\pi$	0		
	$+$	$+$	
0	∞	......	asymp. $y = -a\pi^2$
	$-$	$+$	
π	0		
	$+$	$+$	

θ	r
$\frac{\pi}{2}$	$-\frac{3\pi}{2}a$
$\frac{3\pi}{2}$	$\frac{5\pi}{6}a$
....	....

Hence the curve has the form represented in fig. 29; the dotted part corresponding to the negative values of θ, and

$$SB = a\pi^2, \quad SC = \frac{3\pi a}{2}, \quad SD = \frac{5\pi a}{6}.$$

A reference to the plates M will help the Student to make out the forms of curves from tables of the corresponding values of the co-ordinates.

The curves whose equations are

$$y^2(x^2 - 4a^2) = x^2(x^2 - a^2) \ldots\ldots (42),$$

$$y^2(x^2 - 4a^2) = (x^2 - a^2) \ldots\ldots (43),$$

$$r = ae^\theta \sin\theta \ldots\ldots (44),$$

$$r = ae^\theta \sin 2\theta \ldots\ldots (45),$$

$$r = ae^\theta \sin\frac{\theta}{2} \ldots\ldots (46),$$

$$r = a\sin 3\theta \ldots\ldots (47),$$

$$r = a\sin\frac{\theta}{2} \ldots\ldots (48),$$

$$r = a\sin\frac{\theta}{3} \ldots\ldots (49),$$

are respectively represented by figures 42, 43, 44, &c...&c.

CHAPTER XVI.

SINGULAR POINTS OF CURVES.

IN the preceding Chapter we shewed how the *general* form of a curve may be traced: we now proceed to shew how its remarkable points may be detected and examined.

Proposition. 227. *To investigate the nature of a curve in the immediate vicinity of any proposed point of it.*

Let the equation to the curve be

$$y = f(x);$$

let the abscissa of the proposed point be $x = a$, and let x_1 be any value of x a little less than a, and x_2 a little greater. Then we shall consider several different cases which may occur.

Case 1. 228. *We shall in the first place suppose that* $\mathrm{f}(\mathrm{x}_1)$ *and* $\mathrm{f}(\mathrm{x}_2)$ *have one real value each, and only one, and this being assumed:*

(1) Let $f(a) = \infty$, $f(x_1) =$ positive, $f(x_2) =$ positive,

then the ordinate belonging to the abscissa a is an asymptote, and the curve lies above the axis of x on each side of it: therefore fig. 50 represents the curve, where AM is the abscissa a, and $MP (= \infty)$ the corresponding ordinate.

If $f(x_1) =$ negative, and $f(x_2) =$ negative, fig. 51 represents the curve.

If $f(x_1) =$ negative and $f(x_2) =$ positive, then the curve lies below the axis of x on the left side of MP, and above it on the right, and therefore fig. 52 represents it.

If $f(x_1)$ = positive, and $f(x_2)$ = negative, fig. 53 represents the curve.

(2) Let $f(a)$ be a finite quantity; take $MP = f(a)$, fig. 54, and draw the line $O'PO$ parallel to AM; then $f(x) - f(a)$ represents the distance of any point on the curve above $O'PO$. Hence

If $f'(a) = \infty$, $f(x_1) - f(a)$ = positive, $f(x_2) - f(a)$ = positive, the tangent at P coincides with MP, but the curve lies above OPO on each side of MP; therefore fig. 54 represents its form.

If $f'(a) = \infty$, $f(x_1) - f(a)$ = negative, and $f(x_2) - f(a)$ = negative; then fig. 55 represents the curve.

If $f'(a) = \infty$, $f(x_1) - f(a)$ = negative, and $f(x_2) - f(a)$ = positive; then the curve touches MP at P, lies below it on the left side of MP, and above it on the right; therefore fig. 56 represents it.

If $f'(a) = \infty$, $f(x_1) - f(a)$ = positive, and $f(x_2) - f(a)$ = negative then fig. 57 represents the curve.

If, however, $f'(a)$ be a finite quantity, draw the line $T'PT$ (fig. 58) making an angle $\tan^{-1} f'(a)$ with the axis of x; then this line is the tangent to the curve at P, and, its equation being $y' - f(a) = f'(a)(x - a)$,

$$y - y' \text{ or } f(x) - f(a) - f'(a)(x - a)$$

is the distance of any point of the curve above this line. Hence, if for brevity, we put

$$\phi(x) = f(x) - f(a) - f'(a)(x - a),$$

it is evident that:

If $\phi(x_1)$ = positive, and $\phi(x_2)$ = positive, the curve lies above the tangent $T'PT$ on both sides of MP, and is therefore represented by fig. 59.

If $\phi(x_1)$ = negative, and $\phi(x_2)$ = positive, the curve lies below $T'PT$ on the left side of MP, and above it on the right, and is therefore represented by fig. 60.

If $\phi(x_1)$ = positive, and $\phi(x_2)$ = negative, fig. 61 represents the curve.

These are all the cases that can occur when y has one real value, and only one, for each value of x.

Case 2. 229. Let us now in the second place suppose that $f(x_1)$ has no real value, and $f(x_2)$ two real values, and only two; and this being assumed:

(1) Let $f(a) = \infty$, and both the values of $f(x_2)$ positive: then MP is an asymptote to two branches of the curve, both on the right side of MP; therefore fig. 62 represents the curve.

If both values of $f(x_2)$ be negative, fig. 63 represents the curve.

If one value be positive, and the other negative, fig. 64.

(2) Let $f(a)$ be a finite quantity, and $f'(a) = \infty$, then:

If both values of $f(x_2) - f(a)$ be positive, two branches of the curve touch MP at P, but do not go below $O'PO$, nor appear on the left side of MP: therefore fig. 65 represents the curve.

If both values be negative, fig. 66.

If one value be positive and the other negative, fig. 67.

(3) Let $f(a)$ be a finite quantity, and $f'(a)$ be so also, then:

If both values of $\phi(x_2)$ be positive, two branches of the curve touch the line $T'PT$ at P, but do not go below it, nor appear on the left side of MP: therefore fig. 68 represents the curve.

If both values be negative, fig. 68, bis.

If one value be positive, and the other negative, fig. 69.

If we suppose that $f(x_2)$ has no real value, and $f(x_1)$ two real values and only two, then it is evident the curve in each case will be exactly similar in form to what it is when $f(x_1)$

has no real value, and $f(x_2)$ two real values; it will be merely reversed in position: thus, instead of fig. 68, we shall have fig. 70; and similarly the other figures.

These are all the cases that can occur when $f(x)$ has two real values, and only two, for some values of x, and no real values for others.

Conjugate points.

230. It sometimes happens that $f(x_1)$ and $f(x_2)$ are both impossible, no matter how near x_1 and x_2 may be taken to a, and yet $f(a)$ a real quantity; in such a case the point P is a point belonging to the curve, since its co-ordinates satisfy the equation to the curve, and we define the curve to be the assemblage of all the points whose co-ordinates satisfy that equation: but no points in the immediate vicinity of P belong to the curve; P therefore is an isolated point of the curve, completely detached from the other points. Such a point is usually called a *conjugate point.*

Multiple points.

231. If $f(x_1)$ and $f(x_2)$ have each several values while $f(a)$ has only one value, then several branches of the curve must meet at the point P; in such a case P is called a *multiple point.*

We shall not extend this enumeration of cases any farther, as it will be easy, after what has been explained, to make out the form of the curve in any case that may present itself.

Examples.

232. The following are examples of the cases that we have just discussed.

$$(1)\quad y = a + \frac{a^3}{(x-a)^2}.$$

Here $f(a) = \infty$, $f(x_1) =$ positive, $f(x_2) =$ positive; therefore fig. 50 represents the curve.

$$(2)\quad y = \frac{x-2a}{(x-a)^2}.$$

Here $f(a) = \infty$, $f(x_1) =$ negative, $f(x_2) =$ negative; (fig. 51).

(3) $y = \dfrac{x^2}{x - a}$.

Here $f(a) = \infty$, $f(x_1)$ = negative, $f(x_2)$ = positive; (fig. 52).

(4) $y = a + a^{\frac{1}{3}}(x - a)^{\frac{2}{3}}$.

Here $f(a) = a$, $f'(a) = \infty$, $f(x_1) - f(a)$ = positive; $f(x_2) - f(a)$ = positive; (fig. 54).

(5) $y = a + a^{\frac{2}{3}}(x - a)^{\frac{1}{3}}$; (fig. 55).

(6) $y = x + \dfrac{(x - a)^2}{a}$.

Here $f(a) = a$, $f'(a) = 1$, $\phi(x_1)$ = positive, $\phi(x_2)$ = positive; (fig. 59).

(7) $y = x + \dfrac{(x - a)^3}{a^2}$; (fig. 60).

(8) $\{y(x - a) - a^2\}^2 = a(x - a)^3$.

Solving this, we have $y = \dfrac{a^2}{x - a} \pm \sqrt{ax - a^2}$.

Here therefore $f(a) = \infty$, $f(x_1)$ is impossible, and $f(x_2)$ has two real values both positive*; (fig. 62).

(9) $(y - a)^2(ax - a^2) = a^4$.

Solving this, $y = a \pm \dfrac{a^2}{\sqrt{ax - a^2}}$.

Here $f(a) = \infty$, $f(x_1)$ is impossible, and $f(x_2)$ has two real values, one positive and the other negative; (fig. 64).

(10) $(y - a)^2 = ax - a^2$.

Solving this $y = a \pm \sqrt{ax - a^2}$,

---

* For when x nearly $= a$ the first term is very great compared with the second, and therefore y has the same sign as the first term.

$$\text{and } \frac{dy}{dx} = \pm \frac{a}{2\sqrt{ax - a^2}}.$$

Here $f(a) = a$, $f'(a) = \infty$, $f(x_1) - f(a)$ is impossible, and $f(x_2) - f(a)$ has two real values, one positive and the other negative; (fig. 67).

(11) $a(y - x)^2 = (x - a)^3$.

Solving this $y = x \pm \dfrac{(x - a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}$,

$$\text{and } \frac{dy}{dx} = 1 \pm \tfrac{3}{2}\left(\frac{x - a}{a}\right)^{\frac{1}{2}}.$$

Here $f(a) = a$, $f'(a) = 1$, $\phi(x_1)$ is impossible, and $\phi(x_2)$ has two real values, one positive and the other negative; (fig. 69).

(12) $y = x + a^{\frac{2}{3}}(x - a)^{\frac{1}{3}} \pm a^{\frac{1}{2}}(x - a)^{\frac{1}{2}}$.

Here $f(a) = a$, $f'(a) = \infty$, $f(x_1)$ is impossible;

$$f(x) - f(a) = (x - a)^{\frac{1}{3}}\left\{a^{\frac{2}{3}} \pm a^{\frac{1}{2}}(x - a)^{\frac{1}{6}}\right\}.$$

Now $(x_2 - a)^{\frac{1}{6}}$ is very small, and $\therefore$ $a^{\frac{2}{3}} \pm a^{\frac{1}{2}}(x_2 - a)$ is positive whether we take the + or the −. Hence $f(x_2) - f(a)$ has two real values both positive; (fig. 65).

(13) $y = x + \dfrac{(x - a)^2}{a} \pm \dfrac{(x - a)^{\frac{5}{2}}}{a^{\frac{3}{2}}}$.

Here $f(a) = a$, $f'(a) = 1$, $f(x_1)$ is impossible;

$$\phi(x) = \frac{(x - a)^2}{a}\left\{1 \pm \frac{(x - a)^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right\}.$$

Now $(x_2 - a)^{\frac{1}{2}}$ is very small, and $\therefore$ $1 \pm \dfrac{(x_2 - a)^{\frac{1}{2}}}{a^{\frac{1}{2}}}$ is positive whether we take the + or the −; hence $\phi(x_2)$ has two real values both positive; (fig. 68).

(14) $y = a \pm (x - a)\sqrt{x - 2a}$.

Here $f(a) = a$, and $f(x_1)$ $f(x_2)$ are both impossible, since $\sqrt{x - 2a}$ is impossible for all values of x less than $2a$;

hence the point whose co-ordinates are $x = a$, $y = a$ is a conjugate point of this curve.

(15) $\{a(y-a) - (x-a)^2\}^2 = a^2(x-a)^2$.

Solving this $y = a \pm (x-a) - \dfrac{(x-a)^2}{a}$.

Here $f(x)$ has two different real values for every value of x except $x = a$; hence the curve has two branches which meet at the point whose co-ordinates are $x = a$, $y = a$. Since $f'(a) = \pm 1$ the tangents of the two branches at the point P make angles 45^0 and -45^0 with the axis of x; the branches therefore cross each other at right angles at the point P; the concavities of both turned downwards since $f^2(a)$ is negative; (fig. 71).

(16) $y = x + (x-a)\sin^{-1}\dfrac{x}{a}$. Here an infinite number of branches cross at the point ($x = a$, $y = a$).

Cusps.

233. The curve at the point P in (fig. 54, 65) is called a *cusp* from its pointed form: in (fig. 69) it is called a cusp of the first kind, and in (fig. 68) a cusp of the second kind.

Singular points.

Points of contrary flexure, cusps, conjugate points, multiple points, &c. are called *singular points.*

How the existence of each of these singular points is indicated.

234. *The existence of a point of contrary flexure is indicated by* $\dfrac{d^2y}{dx^2}$ *changing its sign.* (*See* 180).

235. *The existence of a cusp is indicated by the curve not crossing a right line which does not coincide with the tangent.*

If $\dfrac{dy}{dx}$ be a finite quantity at a certain point, and if the curve does not cross the ordinate at that point; or, if $\dfrac{dy}{dx} = \infty$ at a certain point, and the curve does not cross the line drawn through that point parallel to the axis of x; then there is a cusp at that point.

How to determine whether the point is a cusp of the first kind, or not.

236. To determine whether the cusp is of the first or second kind we must find $\phi(x)$ (see 228), and arrange it in ascending powers of $x - a$; suppose that the result of this is

$$\phi(x) = A(x-a)^m + B(x-a)^n + \&c.$$
$$= (x-a)^m \{A + B(x-a)^{n-m} + \&c.\}$$

Now when $x - a$ is very small, $A + B(x-a)^{n-m} + \&c.$ will have the same sign as A, therefore $\phi(x_2)$ will have the same sign as $A(x_2 - a)^m$. Hence if m be a fraction with an even denominator (of course reduced to its lowest terms) $A(x_2 - a)^m$ will have two real values with opposite signs, and therefore the two values of $\phi(x_2)$ have opposite signs, and therefore the cusp is of the first kind. But if m be an integer or a fraction with an odd denominator $A(x_2 - a)^m$ has only one real value, and therefore both values of $\phi(x_2)$ have the same sign as A, and consequently the cusp is of the second kind represented by fig. 68, or 68 bis, according as A is positive or negative.

Of course we here suppose that $\phi(x_2)$ has two and only two real values, and $\phi(x_1)$ no real value. If it be $\phi(x_1)$ which has the two real values, all that we have just said is equally true; only the cusp in each case will be reversed in position.

Cusps, when $f'(a) = \infty$, how distinguished.

We here suppose also that $f'(a)$ is not infinite. If it be, the cusp will be of the first kind, if $f(x_1) - f(a)$ and $f(x_2) - f(a)$ have each one real value of the same sign; and a cusp of the second kind, if $f(x_2) - f(a)$ has two real values, and $f(x_1) - f(a)$ no real value, or the reverse.

If $f(a)$, $f'(a)$, $f^2(a)$, &c. ......have each one real finite value, the point $\{x=a, y=f(a)\}$ cannot be a singular

237. If the quantities $f(a)$, $f'(a)$, $f^2(a)$, &c. have each one real and finite value; then when x is taken sufficiently near to a, we have

$$f(x) = f(a) + f'(a)\frac{x-a}{\Gamma 1} + f^2(a)\frac{(x-a)^2}{\Gamma 2} + \ldots \textit{ ad infinitum},$$

which gives one real value for each value of x.

point of any kind except one of contrary flexure.

Under such circumstances we have therefore the case where $f(a)$ and $f'(a)$ are both finite, and $f(x_1)$ and $f(x_2)$ have each one real value: the curve therefore must assume a form similar to one of those in fig. 72 in the immediate vicinity of the point P: consequently P may or may not be a point of contrary flexure, but it cannot be a cusp, or a multiple point, or a conjugate point.

Hence if $\mathrm{f(a)}$, $\mathrm{f'(a)}$, $\mathrm{f^2(a,)}$ *&c. have each one real and finite value, the point whose co-ordinates are* a *and* $\mathrm{f(a)}$ *must be either an ordinary point or a point of contrary flexure: it cannot be a singular point of any kind except one of contrary flexure.*

At all singular points except one of contrary flexure two conditions hold, when the equation to the curve is algebraical and cleared of radicals.

238. Suppose that the equation to the curve is an *algebraical* equation between x and y *cleared of radicals* of the form

$$A + Ba + Cy + Dx^2 + Exy + Fy^2 + Gx^3 \ldots + Py^p = 0,$$

which for brevity we shall represent by

$$U = 0 \ldots\ldots (1);$$

then to find the successive differential coefficients of y we have the following equations got by differentiating (1), viz.

$$d_y U \,.\, \frac{dy}{dx} + d_x U = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$d_y U \,.\, \frac{d^2y}{dx^2} + d_y^2 U \,.\, \left(\frac{dy}{dx}\right)^2 + 2 d_x d_y U \,.\, \frac{dy}{dx} + d_x^2 U = 0 \ldots (3),$$

$$d_y U \,.\, \frac{d^3y}{dx^3} + d_y^3 U \,.\, \left(\frac{dy}{dx}\right)^3 + \&\text{c}\ldots = 0 \ldots\ldots (4),$$

&c. ...... &c. ......

....

From these equations we may find the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ ... &c. corresponding to any values a and b of x and y which satisfy (1): for by substituting a and b for x and y

in (2), (3), (4), &c. we find the corresponding value of $\frac{dy}{dx}$ from (2), and then that of $\frac{d^2y}{dy^2}$ from (3), and then that of $\frac{d^3y}{dx^3}$ from (4), and so on.

Now it is evident from the form of the equations (1), (2), (3), (4), &c. that if d_yU is not zero, we thus obtain a single finite value for each of the quantities $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ .... for then we evidently find

$$\frac{dy}{dx} = \frac{\text{a quantity which cannot be infinite}^*}{d_yU}$$

$$\frac{d^2y}{dx^2} = \frac{\text{a quantity which cannot be infinite}}{d_yU},$$

and so on.

Hence it appears by the preceding Article that if d_yU be not zero, the point (ab) must be either an ordinary point or a point of contrary flexure. And the same may be proved in exactly the same manner of d_xU.

Hence at all singular points except a point of contrary flexure we must have

$$d_xU = 0, \quad d_yU = 0,$$

$U = 0$ *being an algebraical equation cleared of radicals.*

Hence we may find all the points of a curve which may be singular.

239. If therefore the equation to a curve can be reduced to the form of a rational and integral algebraical equation $U = 0$, and if we determine those points whose co-ordinates satisfy this equation, and moreover the equations $d_xU = 0$, $d_yU = 0$; then no other points of the curve but these can be singular points of any kind, except points of contrary flexure.

To determine whether such points are really singular points, and the class to which they belong to, we must examine them separately; and first, we must determine how many branches of the curve meet at each of these points, which we may do as follows.

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* The numerators of these fractions cannot be infinite, because U is a rational and integral function of x and y.

To determine how many branches of a curve pass through a point at which $d_xU=0$, $d_yU=0$.

240. Let $x=a$ and $y=b$ be the co-ordinates of any one of these points; then since the substitution of a and b for x and y makes

$$d_xU=0 \text{ and } d_yU=0,$$

we must proceed to determine $\frac{dy}{dx}$ as in (153, &c.), and we shall arrive at an equation for determining $\frac{dy}{dx}$ of a higher order than the first. If this equation has n real roots, then n branches of the curve and no more meet at the point (ab). If any of these roots be equal, then the corresponding branches have the same tangent at the point (ab). If when x is a little greater than a, y has n real values, and when x is a little less than a only $n-2m$ real values, (for an *odd* number of real roots cannot disappear) then $2m$ of the branches do not appear on the left side of the ordinate b, and each two of these branches must therefore either touch the ordinate b as in fig. 67, which will be the case with those branches for which $\frac{dy}{dx}$ becomes ∞ when $x=a$; or else they must meet the point (ab) in the form of a cusp as in (fig. 68) or (fig. 69), which will be the case with those branches for which $\frac{dy}{dx}$ does not become infinite when $x=a$.

If the $2m$ real values of y are wanting when x is a little greater than a, the same is true, only the cusps, &c. are reversed in position.

If the equation for determining $\frac{dy}{dx}$ have no real root (ab) is a conjugate point.

The method of expansion in (145) will often en-

241. We may often very readily find the nature of a curve near a proposed point (ab) by expanding $y-b$ in powers of $x-a$, by the method given in (145): each different expansion

will indicate a different branch of the curve, and will shew how it lies near the proposed point.

able us to make out the nature of a curve near a proposed point.

We may also by the same method make out the nature of the infinite branches of a curve, by putting $\frac{1}{z}$ for x, and expanding y in powers of z.

And thus we may often make out very readily the form of a curve when we cannot solve its equation. (For Examples see Appendix V.)

CHAPTER XVII.

THE GENERAL THEORY OF CONTACT. INTERSECTION OF CONSECUTIVE CURVES.

Different orders of contact.

242. Let PQ, PQ', PQ'', (fig. 30) be three curves having a common point P, the co-ordinates of which are $AM = a$, $MP = b$; and let the respective equations to these three curves be

$$NQ = y = f(x), \quad NQ' = y' = \phi(x), \quad NQ'' = y'' = \psi(x);$$

$$\text{then } \frac{QQ''}{QQ'} = \frac{\psi(x) - f(x)}{\phi(x) - f(x)}.$$

Which, since the three curves meet at P, and therefore

$$f(a) = \phi(a) = \psi(a),$$

assumes the form $\frac{0}{0}$ when $x = a$.

Hence, by (148), when x approaches a, we have in general,

$$\text{lim. val. of } \frac{QQ''}{QQ'} = \frac{\psi'(a) - f'(a)}{\phi'(a) - f'(a)}.$$

Now suppose that $\phi'(a) = f'(a)$, while $\psi'(a)$ does not $= f'(a)$; then this limiting value is infinite; therefore when the point N approaches M, QQ'' gets continually larger in comparison with QQ', and may be made as many times larger than it as we please by sufficiently diminishing MN; and therefore the curve PQ' must lie infinitely closer to PQ in the immediate vicinity of the point P than the curve PQ'' does.

Hence it appears that if, for the same abscissa, two curves have the same value, not only of y, but also of $\frac{dy}{dx}$, they not only meet, but also lie infinitely closer to each other in the immediate vicinity of the point of occurse than they would do if they had not the same value of $\frac{dy}{dx}$ at that point.

But suppose that $\psi'(a) - f'(a)$, and $\phi'(a) - f'(a)$ are both zero, then by (148) we have in general, when x approaches a,

$$\text{lim. val. of } \frac{QQ''}{QQ'} = \frac{\psi^2(a) - f^2(a)}{\phi^2(a) - f^2(a)}.$$

Suppose here that $\phi^2(a) = f^2(a)$ while $\psi^2(a)$ does not $= f^2(a)$, then this limiting value is infinite. We may therefore conclude just as before, that if for the same abscissa two curves have the same values, not only of y and $\frac{dy}{dx}$, but also of $\frac{d^2y}{dx^2}$, then they lie infinitely closer to each other in the immediate vicinity of the point of occurse than they would do if they had not the same value of $\frac{d^2y}{dx^2}$.

And in general, if

$$\phi(a) - f(a), \quad \phi'(a) - f'(a) \ldots\ldots \phi^{n-1}(a) - f^{n-1}(a),$$
$$\psi(a) - f(a), \quad \psi'(a) - f'(a) \ldots\ldots \psi^{n-1}(a) - f^{n-1}(a),$$

be each zero, we have, when x approaches a,

$$\text{lim. val. of } \frac{QQ''}{QQ'} = \frac{\psi^n(a) - f^n(a)}{\phi^n(a) - f^n(a)}.$$

Suppose here that $\phi^n(a) = f^n(a)$ while $\psi^n(a)$ does not $= f^n(a)$; then this limiting value is infinite. We may therefore conclude *that if for the same abscissa two curves have the same values of* y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2} \cdots \frac{d^ny}{dx^n}$, *then they lie infinitely closer to each other in the immediate vicinity of the point of occurse than they would do if they had not the same value of* $\frac{d^ny}{dx^n}$.

Orders of contact depend upon the differential coefficients.

Hence it is that when two curves meet, and at the point of occurse have the same value of $\frac{dy}{dx}$, they are said to have contact of the first order, if moreover the same value $\frac{d^2y}{dx^2}$, contact of the second order, and so on; and in general, if they have the same values of $\frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^ny}{dx^n}$, they are said to have *contact of the* n^th^ *order*.

Contact of an even order is accompanied with intersection, whereas contact of an odd order is not.

243. If $\phi(a) - f(a)$, $\phi'(a) - f'(a) \dots \phi^n(a) - f^n(a)$ be each zero, and $\phi^{n+1}(a) - f^{n+1}(a)$ not zero, then by (121), $\phi(x) - f(x)$ has the same sign as

$$\{\phi^{n+1}(a) - f^{n+1}(a)\}\,(x-a)^{n+1}$$

for all values of x sufficiently near a: now if the contact be of an even order n is even, and $n+1$ is odd, and therefore $\phi(x) - f(x)$ changes its sign when x passes through the value a; therefore, at one side of the point P, $\phi(x)$ is greater than $f(x)$, and at the other side less; i. e. the curves cross each other at P, as in fig. 31. But if the contact be of an odd order, n is odd, and $n+1$ even, and therefore QQ' does not change its sign when x passes through the value a; i. e. the curves meet each other without crossing at the point P, as in fig. 32. *Hence contact of an even order is accompanied with intersection, but contact of an odd order is not.*

What has been said is true for oblique and polar co-ordinates also.

244. It is evident that all we have just said is equally true whether the co-ordinates be rectangular or oblique. Moreover it is easy to see that if we refer to curves to any new axes of co-ordinates, the degree of contact is not altered.

The order of contact is not changed by a change of co-ordinates.

It is clear also that what we have said applies equally to polar co-ordinates; i. e. if $\frac{dr}{d\theta}, \frac{d^2r}{d\theta^2} \dots \frac{d^nr}{d\theta^n}$ have the same values in both curves at the point of occurse, the contact will be of the n^{th} order: for then it is easy to see, putting $x = r\cos\theta$, $y = r\sin\theta$, that $\frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^ny}{dx^n}$ will also have the same values in both curves.

Example 1. A right line.

245. To make a line whose equation is

$$y_1 = Ax + B \text{ .... } (1),$$

have contact of the first order with a given curve whose equation is $y = f(x)$, at a given point whose abscissa is a.

y must $= y_1$ when $x = a$; therefore we have

$$f(a) = Aa + B \text{ ...... } (2).$$

$\frac{dy}{dx}$ must $= \frac{dy_1}{dx}$ when $x = a$; therefore we have

$$f'(a) = A \text{ .... } (3).$$

(1), (2) and (3) give

$$y - f(a) = f'(a)(x - a),$$

which is the equation required. It coincides with the common equation to the tangent, see (173), and hence the ordinary tangent has contact of the first order with the curve.

If we wish to make the line (1) have contact of the second order with the curve, we must put $\frac{d^2y}{dx^2} = \frac{d^2y_1}{dx^2}$ when $x = a$: but this gives $f^2(a) = 0$, and this equation of course cannot hold except at particular points of the curve. Hence a right line cannot have contact of an order higher than the first with a curve except at particular points of it, determined by the equation $\frac{d^2y}{dx^2} = 0$ (or in some cases $\frac{d^2y}{dx^2} = \infty$, see note *).

Example 2. A right line referred to polar co-ordinates.

246. Suppose that the curve is referred to polar co-ordinates, and that its equation is $r = f(\theta)$. The equation to a right line being $y = \tan\alpha\,(x + a)$, putting $x = r_1\cos\theta$, $y = r_1\sin\theta$, becomes

$$r_1 = \frac{a\sin\alpha}{\sin(\theta - \alpha)},$$

---

* When $\frac{d^2y}{dx^2} = \infty$ $\frac{d^2x}{dy^2}$ (y being the independant variable) will be 0 in certain cases, and then there will be contact of the second order.

$$\text{then } \frac{1}{r_1} = \frac{\sin(\theta - \alpha)}{a \sin \alpha}, \quad \frac{1}{r_1^2}\frac{dr_1}{d\theta} = \frac{\cos(\theta - \alpha)}{a \sin \alpha},$$

$$\text{and } \therefore \frac{dr_1}{d\theta} = -r_1 \cot(\theta - \alpha).$$

Now suppose that the line has contact of the first order with the curve at the point whose polar abscissa is $\theta = \beta$, then when $\theta = \beta$ we must have $r_1 = r$ and $\frac{dr_1}{d\theta} = \frac{dr}{d\theta}$,

$$\therefore f(\beta) = \frac{a \sin \alpha}{\sin(\beta - \alpha)}, \quad \text{and } f'(\beta) = -f(\beta)\cot(\beta - \alpha).$$

From the second of these equations we may determine α, and then from the first a, and thus the polar equation to a line touching a curve given by a polar equation at any proposed point may be found.

Since β may have any value we have in general

$$f'(\theta) = -f(\theta)\cot(\theta - \alpha),$$

$$\text{or} \quad \tan(\alpha - \theta) = \frac{f(\theta)}{f'(\theta)} = \frac{r\,d\theta}{dr},$$

which coincides with the result obtained in (196). Since $\alpha - \theta$ is evidently the angle under the tangent and radius vector, i.e. ϕ.

The order of contact which we may make two curves have with each other.

247. The order of contact which we may make two curves have with each other depends in general upon the number of constants we have at our disposal; if there be $n+1$ constants disposable we may so determine them that the contact shall be of the n^{th} order; for then we may, by giving them proper values, satisfy the $n+1$ equations

$$y_{,} = y, \quad \frac{dy_{,}}{dx} = \frac{dy}{dx}, \quad \frac{d^2y_{,}}{dx^2} = \frac{d^2y}{dx^2} \cdots \frac{d^ny_{,}}{dx^n} = \frac{dy^n}{dx^n},$$

which equations must be fulfilled to produce contact of the n^{th} order.

Thus in the most general equation to the right line, $y = Ax + B$, there are two disposable constants A and B, and therefore we may in general make a right line have contact of the first order, but no higher, with a given curve at a given point of it. In the most general equation to a circle, viz. $(x-A)^2+(y-B)^2=R^2$ there are three disposable constants A, B and R, and therefore we may in general make a circle have contact of the second order, but no higher, with a given curve at a given point.

Example 3. A circle made to touch a proposed curve with contact of 2nd order.

248. To determine A, B, R, so that the circle

$$\{(x-A)^2+(y_1-\beta)^2=R^2\}$$

shall have contact of the second order with the curve

$$\{y=f(x)\}$$

at the point whose abscissa is a.

We have

$$(x-A)^2+(y_1-B)^2=R^2 \ldots (1),$$

$$x-A+(y_1-B)\frac{dy_1}{dx}=0 \quad \ldots (2),$$

$$1+\left(\frac{dy_1}{dx}\right)^2+(y_1-B)\frac{d^2y_1}{dx^2}=0 \quad \ldots (3).$$

Now since the circle touches the curve with contact of the second order at the point whose abscissa is a, if for brevity we put $f(a)=b$, $f'(a)=p$, $f^2(a)=q$ must have

$$y_1=b, \quad \frac{dy_1}{dx}=p, \quad \frac{d^2y_1}{dx^2}=q \quad \text{when } x=a,$$

and therefore putting $x=a$ in the equations (1), (2), (3), they become

$$(a-A)^2+(b-B)^2=R^2 \ldots (1),$$

$$a-A+(b-B)\,p=0 \quad \ldots (2),$$

$$1+p^2+(b-B)\,q=0 \quad \ldots (3).$$

$$(3) \text{ gives } b-B=-\frac{1+p^2}{q},$$

$$\text{then (2) gives } a - A = p\,\frac{1+p^2}{q},$$

$$\text{and then (1) gives } R = \pm\,\frac{(1+p^2)^{\frac{3}{2}}}{q},$$

which equations give us B, A, and R, and thus the circle is completely determined.

Circle of curvature. Its radius is the index of curvature ρ. See (181.)

This circle is commonly called the osculating circle, or the circle of curvature. Since a may belong to any point whatever of the curve, we may put x instead of a, and ∴ y, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ instead of b, p, q respectively, and then we have

$$R = \pm\,\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \pm\,\frac{ds^3}{dx\,d^2y};$$

hence it appears that $R = \rho$ (see 184), and hence the point P' (185) is the center of the osculating circle.

It follows therefore that the limiting position of the intersection of two consecutive normals to a curve when they approach each other is the center of the osculating circle.

We may also arrive at this result by means of the following article.

Of the ultimate intersection of curves.

249. Let the equations to two curves be

$$f(xya) = 0 \ldots\ldots (1),$$

$$f(xya') = 0 \ldots\ldots (2),$$

(2) differing from (1) only in having a' in place of a, a and a' being two constants to which we may assign any values we please. Then if we determine x and y from these two equations taken together, the resulting values will be the co-ordinates of a point of intersection of the two curves. Now this point, which we shall call P, must be some definite point as long as a' is different from a, but if a' becomes

equal to a, it ceases to be a definite point, since then the two curves are identical. x and y therefore become illusory when $a' = a$, but of course they have some limiting value when a' approaches a as in the example (22). We may determine the limiting values of x and y as follows.

By Lemma XXI. the equation (2) may be put in the form

$$f(xya) + \{f'(xy\underset{\cdot}{a}) + Q\}(a' - a) = 0, \quad \text{(see 77)}$$

where Q is a quantity which may be diminished *ad libitum* by sufficiently diminishing $a' \sim a$. In virtue of (1), and dividing out $a' - a$, this equation becomes

$$f'(xy\underset{\cdot}{a}) + Q = 0 \ldots\ldots (3).$$

Of course we necessarily suppose in this process that a' does not actually $= a$.

From (1) and (3) we may determine x and y, which will of course be different for different values of a'. Let a' approach a; then, in virtue of (1) and (3), the limiting values of $f(xya)$ and of $f'(xy\underset{\cdot}{a}) + Q$ must be zero; if therefore x_1 and y_1 be the limiting values of x and y, since that of Q is zero, we have by Lemma VIII.

$$f(x_1y_1a) = 0, \text{ and } f'(x_1y_1\underset{\cdot}{a}) = 0,$$

and from these equations we may determine x_1 and y_1.

Since x_1 and y_1 are the limiting values of x and y when a' approaches a, it is clear that we may diminish the distance between the points (xy) and (x_1y_1) *ad libitum* by sufficiently diminishing $a' - a$. The point x_1y_1 is therefore called the *ultimate intersection* of two consecutive positions of the curve (1), and its co-ordinates are the values of x and y which are obtained from the equation

$$f(xya) = 0,$$

and its partial derivative with respect to a, viz.

$$f'(xy\underset{\cdot}{a}) = 0;$$

a is called the variable parameter of the curve.

Example 1. **250.** To find the ultimate intersection of two consecutive positions of a line which always includes equal areas between it and the co-ordinate axes.

Let $\frac{x}{\alpha} + \frac{y}{\beta} = 1$ be the equation to the line, and $2c$ the constant area; then $\alpha\beta = c$, and the equation to the line becomes

$$\frac{x}{\alpha} + \frac{\alpha y}{c} - 1 = 0 \ldots\ldots (1),$$

α being the variable parameter.

The partial derivative of this with respect to α is

$$-\frac{x}{\alpha^2} + \frac{y}{c} = 0 \ldots\ldots (2),$$

from these equations we easily get

$$\frac{x}{\alpha} + \frac{x}{\alpha} - 1 = 0, \quad \text{or } x = \frac{\alpha}{2};$$

$$\text{and then } y = \frac{c}{2\alpha}.$$

The value $x = \frac{\alpha}{2}$ shews that the line is bisected by the point of ultimate intersection.

We may verify the correctness of this result thus.

Let α' be another value of α, then the equation to the corresponding line is

$$\frac{x}{\alpha'} + \frac{\alpha' y}{c} - 1 = 0 \ldots\ldots (3),$$

(3) $\alpha' -$ (1) α gives

$$(\alpha'^2 - \alpha^2)\frac{y}{c} - (\alpha' - \alpha) = 0, \quad \text{or } y = \frac{c}{\alpha' + \alpha};$$

and therefore the limiting value of y when α' approaches α is $\frac{c}{2\alpha}$, and therefore by (1) that of x is $\frac{\alpha}{2}$.

Example 2.

251. Hence we may find the ultimate intersection of two consecutive normals as follows.

The equation to the normal to a curve $y = f(x)$ at the point whose abscissa is a is

$$f'(a)\{y - f(a)\} + x - a = 0 \ldots\ldots\ldots\ldots (1),$$

taking its partial derivative with respect to a we have

$$f^2(a)\{y - f(a)\} - \{f'(a)\}^2 - 1 = 0 \ldots\ldots (2),$$

and the values of x and y got from these equations are the co-ordinates of the point required. These values are

$$y = f(a) + \frac{\{f'(a)\}^2 + 1}{f^2(a)},$$

$$x = a - f'(a) \cdot \frac{\{f'(a)\}^2 + 1}{f^2(a)}.$$

Comparing these values with those of A and B in (249) we see that the point of ultimate intersection of two consecutive normals is the center of the circle of curvature.

How to find the locus of the points of ultimate intersection of a set of curves.

252. We may find the locus of all the points of ultimate intersection of the set of curves represented by the equation

$$f(xya) = 0,$$

by eliminating a between this equation and its partial derivative

$$f'(xya) = 0.$$

Example.

Let us take the first example, last article, in which

$$f(xya) = \frac{x}{a} + \frac{ay}{c} - 1 = 0,$$

$$f'(xya) = -\frac{x}{a^2} + \frac{y}{c} = 0.$$

Eliminating a, we find

$$a = 2x, \quad \text{and } \therefore \ xy = c,$$

which gives a rectangular hyperbola referred to its asymptotes.

CHAPTER XVIII.

LAGRANGE'S THEOREM. ELIMINATION OF CONSTANTS AND FUNCTIONS BY DIFFERENTIATION. EXPANSION BY MEANS OF THIS ELIMINATION.

THE following theorem, due to Lagrange, is often found useful in expanding functions which are given by equations.

Lagrange's Theorem.

253. *Suppose that* u, y, z, x *are variables connected by the equations*

$$\mathrm{u} = \mathrm{f}(\mathrm{y}) \ldots\ldots (1), \qquad \mathrm{y} = \mathrm{z} + \mathrm{x}\phi(\mathrm{y}) \ldots\ldots (2),$$

to expand u *in a series of powers of* x.

To do this, we shall find $d_x{}^n u$ as follows:

y is a function of the independant variables z and x, and we have, by differentiating (2) with respect to z and x,

$$d_x y = \phi(y) + x\phi'(y)\, d_x y; \quad \text{and } \therefore\ d_x y = \frac{\phi(y)}{1 + x\phi'(y)},$$

$$d_z y = 1 + x\phi'(y)\, d_z y; \quad \text{and } \therefore\ d_z y = \frac{1}{1 + x\phi'(y)} \ldots (3);$$

$$\text{hence, } d_x y = \phi(y)\, d_z y \ldots\ldots (4);$$

$$\text{and } \therefore\ d_x u = f'(y)\, d_x y = f'(y)\,\phi(y)\, d_z y \ldots\ldots (5).$$

Now if $\psi(y)$ denote any function of y,

$$\begin{aligned} d_x \{\psi(y)\, d_z y\} &= \psi'(y)\, d_x y\, d_z y + \psi(y)\, d_x d_z y \\ &= d_z \{\psi(y)\, d_x y\} \\ &= d_z \{\psi(y)\,\phi(y)\, d_z y\}, \text{ by } (4). \end{aligned}$$

Hence supposing that $\psi(y) = f'(y)\,\phi(y)$, and differentiating (5), we have

$$d_x^2 u = d_z[f'(y)\{\phi(y)\}^2 d_z y],$$

and again $d_x^3 u = d_z d_x[\ \ldots\ldots\ldots\ldots\ldots\ ]$

$$= d_z^2[f'(y)\{\phi(y)\}^3 d_z y],\text{ similarly,}$$

and so on; and finally,

$$d_x^n u = d_z^{n-1}[f'(y)\{\phi(y)\}^n d_z y].$$

Now when $x = 0$, $z = y$, and by (3) $d_z y = 1$; therefore, when $x = 0$, we have

$$d_x^n u = d_z^{n-1}[f'(z)\{\phi(z)\}^n].$$

Hence, by Taylor's series, if for brevity we put $f'(z) = V$, $\phi(z) = Z$, we have $\{$since $u = f(z)$ when $x = 0\}$,

$$u = f(z) + VZ \cdot \frac{x}{\Gamma 1} + d_z(VZ^2) \cdot \frac{x^2}{\Gamma 2} + d_z^2(VZ^3)\frac{x^3}{\Gamma 3} \cdots$$

$$+ d_z^{n-1}(VZ^n)\frac{x^n}{\Gamma n} + \&c. \ldots$$

which is the developement required. The following example will shew the use of it.

Let $u = e^y$, $y = z + xe^y$. Example.

$$\text{Here } f(y) = \epsilon^y; \quad \therefore f'(z) \text{ or } V = \epsilon^z,$$

$$\phi(y) = e^y; \quad \therefore \phi(z) \text{ or } Z = e^z;$$

$$\therefore d_z^{n-1}(VZ^n) = d_z^{n-1}\{\epsilon^{(n+1)z}\},$$

$$= (n+1)^{n-1}\epsilon^{(n+1)z},$$

which $= 2^0\epsilon^{2z}$, $3\epsilon^{3z}$, $4^2\epsilon^{4z}$, $5^3\epsilon^{5z} \ldots$ when $n = 1, 2, 3, 4, \ldots$ &c. respectively; hence

$$u = e^z + e^{2z}\frac{x}{\Gamma 1} + 3e^{3z}\frac{x^2}{\Gamma 2} + 4^2 e^{4z}\frac{x^3}{\Gamma 3} + 5^3 e^{5z}\frac{x^4}{\Gamma 4} \cdots + \&c.$$

254. When we are given an equation between x and y containing any constants $a_1, a_2, a_3 \ldots a_n$ suppose, we may eliminate these constants by differentiation, as follows. Let the given equation be Elimination of constants by differentiation.

$$U = 0 \ldots\ldots\ldots (1),$$

and let it be differentiated n times, and so give the following equations; viz.

$$dU = 0 \,\ldots\ldots\ldots\, (2),$$

$$d^2U = 0 \,\ldots\ldots\ldots\, (3),$$

$$\ldots\ldots\ldots$$

$$d^nU = 0 \,\ldots\ldots\ldots\, (n+1);$$

then from these $n+1$ equations we may in general eliminate the n quantities $a_1, a_2, a_3 \ldots a_n$, and so obtain an equation between x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2} \ldots \frac{d^ny}{dx^n}$, not containing any of these constants.

Example.

Let the given equation be

$$y - ax^2 + abx = 0 \,\ldots\ldots\ldots\, (1),$$

differentiating twice, we have

$$p - 2ax + ab = 0 \,\ldots\ldots\ldots\, (2), \qquad \left\{ p = \frac{dy}{dx}, \right.$$

$$q - 2a = 0 \,\ldots\ldots\ldots\ldots\ldots\ldots\, (3); \qquad \left| \, q = \frac{d^2y}{dx^2}. \right.$$

hence, $2a = q$, and therefore, by (2),

$$ab = qx - p;$$

and therefore, by (1),

$$y - \frac{qx^2}{2} + qx^2 - px = 0,$$

$$\text{or} \quad y + \frac{qx^2}{2} - px = 0;$$

which is an equation between x, y, p, and q, not containing the constants a and b.

A number of different equations may be formed in this way.

255. It is evident that there be $n+m$ constants a_1, $a_2 \ldots a_n$, $a_{n+1} \ldots a_{n+m}$ in the equation $U=0$, we may eliminate any n of them we please between the equations

$$u = 0,$$

$$du = 0,$$

$$d^2u = 0,$$

$$\dots\dots\dots$$

$$d^nu = 0.$$

Now there are $\dfrac{\overline{m+n}\,.\,\overline{m+n-1}\,\dots\,\overline{m+1}}{1\,.\,2\,\dots\dots\dots\dots\dots\,m}$ (= N suppose) different combinations of the $m+n$ constants a_1, a_2 ... &c.; therefore there are N different ways in which we may form an equation containing x, y, $\dfrac{dy}{dx} \dots \dfrac{d^ny}{dx^n}$ and m of the constants; and consequently we may in general obtain N different equations involving x, y, $\dfrac{dy}{dx} \dots \dfrac{d^ny}{dx^n}$ and m of the constants from the given equation.

In the example just given, if we eliminate a between (1) and (2), we find Example.

$$a = \frac{p}{2x-b} \text{ from (2);}$$

and therefore, by (1),

$$y - \frac{px(x-b)}{2x-b} = 0 \dots\dots (5).$$

Again, if we eliminate b, we obtain

$$y - px + ax^2 = 0 \dots\dots\dots (6),$$

and thus we deduce from (1) two different equations (5) and (6), each involving x, y, $\dfrac{dy}{dx}$, and one of the constants.

256. We may eliminate the constants a_1, a_2, $a_3 \dots a_n$ from $u = 0$ somewhat differently, as follows.

This elimination may be effected somewhat differently.

Solve the equation $u = 0$ for a_1, and let the result be $u_1 = a_1$, then, differentiating, we find

$$du_1 = 0, \text{ in which } a_1 \text{ does not appear.}$$

Solve this equation for a_2, and let the result be $u_2 = a_2$, and then differentiating, we find

$$du_2 = 0, \text{ in which neither } a_1 \text{ nor } a_2 \text{ appear;}$$

and proceeding in this manner we may eliminate a_1, a_2, a_3 ... successively.

Example.

Thus in the example just considered, if we solve for a we find

$$\frac{y}{x^2 - bx} = a,$$

therefore differentiating we have

$$\frac{p\,(x^2 - bx) - y\,(2x - b)}{(x^2 - bx)^2} = 0,$$

$$\text{or}\ \ (x^2 - bx)\,p - y\,(2x - b) = 0,$$

which equation does not contain a.

Again, solving for b, we get

$$\frac{x^2p - 2xy}{xp - y} = b\,;$$

and therefore differentiating we find

$$(x^2q - 2y)\,(xp - y) - (x^2p - 2xy)\,xq = 0,$$

$$\text{or}\quad x^2qy - 2y\,(xp - y) = 0,$$

$$\text{or}\quad x^2y = 2\,(xp - y) = 0,$$

which is the same equation as before.

This method is generally longer than the former.

Eliminations of functions from equations.

257. In the same way that we eliminate constants from equations between two variables x and y, we may eliminate functions from equations between three variables x, y, z.

For let $u = 0$ ... (1)

be an equation between x, y, and z, containing the n functions $\phi_1(v_1)$, $\phi_2(v_2)$, $\phi_3(v_3)$ ... $\phi_n(v_n)$, v_1 v_2 ... v_n being any functions of x y z. Then in virtue of (1) we may consider z as a function of the two independant variables x and y, and we have by successively differentiating with respect to x and y the following set of equations:

$$u = 0,$$

$$d_x u = 0, \quad d_y u = 0,$$

$$d_x^2 u = 0, \quad d_x d_y u = 0, \quad d_y^2 u = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\&c. \ldots$$

$$d_x^m u = 0, \quad d_x^{m-1} d_y u = 0 \ldots\ldots d_y^m u = 0,$$

in all $1 + 2 + 3 \ldots + m$, or $\dfrac{(m+1)m}{2}$ equations. Now in these equations we evidently have involved the functions

$$\begin{array}{llll} \phi_1(v_1), & \phi_2(v_2), & \ldots\ldots\ldots\ldots & \phi_n(v_n), \\ \phi'_1(v_1), & \phi'_2(v_2), & \ldots\ldots\ldots\ldots & \phi'_n(v_n), \\ \phi_1^m(v_1), & \phi_2^m(v_2), & \ldots\ldots\ldots\ldots & \phi_n^m(v_n), \end{array}$$

in all mn different functions.

Hence if $\dfrac{(m+1)m}{2}$ be $> mn$, or $= mn + 1$ at least, we shall be able to eliminate all these functions from the equations, and so form an equation between $x\ y\ z$, $d_x z$, $d_y z$, $d^2_x z$, ... &c. ... $d_x^m z$, $d_x^{m-1} d_y z$ ... $d_y^m z$, containing no trace of the functional letters ϕ_1, ϕ_2, $\phi_3 \ldots \phi_n$

If $n = 1$ the condition $\dfrac{(m+1)m}{2} > mn$ becomes

$$\frac{m+1}{2} > 1, \text{ and } \therefore\ m > 1.$$

If $n = 2$ it becomes

$$\frac{m+1}{2} > 2, \text{ and } \therefore\ m > 3,$$

and so on.

Hence to eliminate one function it will be necessary in general to go beyond the first order of differentiation; to eliminate two functions we must go beyond the third order; and so on. Often however it happens that, in consequence of the peculiar form of the equations, the elimination may be effected without proceeding so far; as the following examples will shew.

Example.

Let the given equation be

$$u = z\phi(x+y) + xy = 0 \ldots (1),$$

then $$d_x u = p\phi(x+y) + z\phi'(x+y) + y = 0 \ldots (2),$$

$$d_y u = q\phi(x+y) + z\phi'(x+y) + x = 0 \ldots (3),$$

where $p = d_x z$,

$q = d_y z$,

(2) and (3) give

$$(p-q)\phi(x+y) + y - x = 0,$$

which by (1) give

$$z\frac{x-y}{p-q} + xy = 0.$$

Thus we obtain from (1), by differentiation as far as the first order, an equation which contains no trace of ϕ.

Example 2.

Let the given equation be

$$\phi(x+z) + \psi(y+z) = 0 \ldots (1),$$

then $$(1+p)\phi'(x+z) + p\psi'(y+z) = 0 \ldots (2),$$

$$q\phi'(x+z) + (1+q)\psi'(y+z) = 0 \ldots (3),$$

(2) and (3) give immediately

$$\frac{1+p}{q} = \frac{p}{1+p} \text{ or } 1+p+q = 0,$$

from which ϕ and ψ are eliminated.

Developement may be effected by eliminating functions. Example cos ($m\cos^{-1}x$)

258. This sort of elimination is often useful in expanding functions, as the following example will shew.

To develope $\cos(m\cos^{-1}x)$ in powers of x, assume

$$y = \cos(m\cos^{-1}x) \ldots (1),$$

then $$\frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}}\sin(m\cos^{-1}x) \ldots (2);$$

$$\therefore \sqrt{1 - x^2}\,\frac{dy}{dx} - m \sin (m \cos^{-1} x) = 0\,;$$

$\therefore$ differentiating again,

$$\sqrt{1 - x^2}\,\frac{d^2 y}{dx^2} - \frac{x}{\sqrt{1 - x^2}}\,\frac{dy}{dx} + \frac{m^2}{\sqrt{1 - x^2}} \cos (m \cos^{-1} x) = 0\,;$$

or multiplying by $\sqrt{1 - x^2}$, and putting y for $\cos (m \cos^{-1} x)$,

$$\frac{d^2 y}{dx^2} - x^2\,\frac{d^2 y}{dx^2} - x\,\frac{dy}{dx} + m^2 y = 0 \ldots (3).$$

Having thus eliminated the functions cos and $\cos^{-1}$ by differentiation, we shall be easily able to develope y in the powers of x by assuming

$$y = A_0 + A_1 x + A_2 x_2 \ldots A_n x^n + A_{n+1} x^{n+1} + A_{n+2} x^{n+2} + \ldots \&c.$$

Substituting this for y in (3) we find for the coefficient of x^n

$$(n + 2)(n + 1) A_{n+2} - n (n - 1) A_n - n A_n + m^2 A_n\,,$$

which being put equal to zero gives

$$A_{n+2} = \frac{n^2 - m^2}{(n + 1)(n + 2)} A_n\,,$$

from which we evidently get

$$A_2 = \frac{- m^2}{\Gamma 2} A_0\,, \qquad A_4 = \frac{(- m^2)(2^2 - m^2)}{\Gamma 4} A_0 \ldots \&c.,$$

$$A_3 = \frac{1^2 - m^2}{\Gamma 3} A_1\,, \qquad A_5 = \frac{(1^2 - m^2)(3^2 - m^2)}{\Gamma 5} A_1 \ldots \&c.$$

To determine A_0 and A_1, put $x = 0$, and $\therefore y = A_0$ and $\frac{dy}{dx} = A_1$, in (1) and (2), and we find

$$A_0 = \cos (m \cos^{-1} 0) = \cos m\,\frac{2r + 1}{2}\,\pi,$$

$$A_1 = m \sin m\,\frac{2r + 1}{2}\,\pi.$$

Hence if for brevity we assume

$$U = 1 - \frac{m^2}{\Gamma 2} x^2 + \frac{m^2 (m^2 - 2^2)}{\Gamma 4} x^4 - \frac{m^2 (m^2 - 2^2)(m^2 - 4^2)}{\Gamma 6} x^6 \ldots \&c.,$$

$$V = m x - \frac{m (m^2 - 1^2)}{\Gamma 3} x^3 + \frac{m (m^2 - 1^2)(m^2 - 3^2)}{\Gamma 5} x^5 \ldots \&c.,$$

we have

$$\cos (m \cos^{-1} x) = U \cos \frac{2r + 1}{2} m\pi + V \sin \frac{2r + 1}{2} m\pi,$$

which is the general expansion of the cosine of a multiple arc in powers of the cosine of the arc.

CHAPTER XIX.

MISCELLANEOUS THEOREMS.

Theorem of Leibnitz. If $y = uv$ to find $d^n(uv)$.

259. If $y = uv$, u and v being two functions of x, then

$$d^n y = d^n u \,.\, v + \frac{n}{1} d^{n-1} u \,.\, dv + \frac{n(n-1)}{\Gamma 2} d^{n-2} u \,.\, d^2 v \ldots + u d^n v.$$

For we have

$$dy = du \,.\, v + u \,.\, dv,$$

$$d^2 y = d^2 u \,.\, v + 2 du \,.\, dv + u d^2 v,$$

$$d^3 y = d^3 u \,.\, v + 3 d^2 u \,.\, dv + 3 du \,.\, d^2 v + u d^3 v,$$

....

Let us assume that

$$d^n y = d^n v \,.\, u + A_n d^{n-1} v \,.\, du + B_n d^{n-2} v \,.\, d^2 u \ldots\ldots$$

then we easily find by differentiating,

$$d^{n+1} y = d^{n+1} v \,.\, u + (A_n + 1) d^n v du + (B_n + A_n) d^{n-1} v d^2 u \ldots;$$

$$\therefore\; A_{n+1} = A_n + 1, \quad B_{n+1} = B_n + A_n, \quad C_{n+1} = C_n + B_n \ldots \text{ \&c.};$$

which shews that if the coefficients of $d^n y$ be the same as those of $(1 + z)^n$ expanded, the same will be true when $n + 1$ is put for n; but we have shewn that this law holds when $n = 1$ or 2 or 3; therefore it is true in general, and we have

$$d^n y = d^n u \,.\, v + \frac{n}{1} d^{n-1} u \,.\, dv + \frac{n(n-1)}{\Gamma 2} d^{n-2} u \,.\, d^2 v \ldots + u d^n v.$$

This is called Leibnitz' Theorem, and is often of use in finding successive differential coefficients.

n^{th} differential coefficient of $\phi\,(a+bx+cx^2)$.

260. *To find the* n^{th} *differential coefficient of* $\mathrm{f\,(x)}$ *if* $\mathrm{f\,(x) = \phi\,(a + bx + cx^2)}$.

We have $f(x+h) = \phi\,(A + Bh + ch^2)$ ... (1),

$$\text{if} \quad A = a + bx + cx^2, \quad B = b + 2cx.$$

Now the general term of $\phi\,(A + Bh + ch^2)$ expanded in powers of $Bh + Ch^2$ is

$$\phi^r\,(A)\,\frac{(Bh + ch^2)^r}{\Gamma r},$$

and the general term of this expanded in powers of ch^2 is

$$\phi^r\,(A)\,.\,\frac{B^{r-s}c^s}{\Gamma s\,.\,\Gamma(r-s)}\,h^{r+s}.$$

In this term let us suppose that $r + s = n$, and it becomes

$$\phi^r\,(A)\,\frac{B^{2r-n}c^{n-r}}{\Gamma\,(n-r)\,\Gamma\,(2r-n)}\,h^n = U_r h^n, \text{ suppose.}$$

Now s is evidently not $> r$, and not < 0, i. e. $n - r$ is not $> r$ and not < 0, i. e. r is not $< \frac{n}{2}$ and not $> n$.

Hence the sum of the terms formed by giving r all possible values is

$$(U_n + U_{n-1} + U_{n-2} \dots U_{\frac{n}{2}})\,h^n \text{ if } n \text{ be even;}$$

$$\text{and } (U_n + U_{n-1} + U_{n-2} \dots U_{\frac{n+1}{2}})\,h^n \text{ if } n \text{ be odd;}$$

and hence it follows that the coefficient of h^n in the second member of (1) expanded in powers of h is

$$U_n + U_{n-1} + U_{n-2} \dots \&\text{c. } U_{\frac{n}{2} \text{ or } \frac{n+1}{2}}.$$

Now the coefficient of h^n in the first member of (1) expanded is, by Taylor's Theorem, $\frac{1}{\Gamma n} f^n(x)$; hence we have

$$f^n(x) = \Gamma n \left(U_n + U_{n-1} + U_{n-2} \dots U_{\frac{n}{2} \text{ or } \frac{n+1}{2}}\right),$$

where

$$U_r = \phi^r(a+bx+cx^2)\,\frac{(b+2cx)^{2r-n}c^{n-r}}{\Gamma(n-r)\,\Gamma(2r-n)},$$

which gives us the differential coefficient required.

261. Let $\phi(a+bx+cx^2) = \sqrt{1+x+x^2} = (1+x+x^2)^{\frac{1}{2}}$, Example of this method.

then $\phi^r(a+bx+cx^2)$

$$= \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)\left(\tfrac{1}{2}-2\right)(\dots)\left(\tfrac{1}{2}-r+1\right)(1+x+x^2)^{\frac{1}{2}-r}$$

$$= \frac{1.3.5\dots(2r-1)}{2^r}(-1)^{r-1}(1+x+x^2)^{\frac{1}{2}-r};$$

$$\therefore\ U_r = \frac{1.3.5\dots(2r-1)}{2^r}(-1)^{r-1}(1+x+x^2)^{\frac{1}{2}-r}\,\frac{(1+2x)^{2r-n}}{\Gamma(n-r)\,\Gamma(2r-n)};$$

$$\therefore\ \Gamma n\, U_n = \frac{1.3.5\dots(2n-1)}{2^n}(-1)^{n-1}(1+x+x^2)^{\frac{1}{2}-n}(1+2x)^n,$$

$$\Gamma n\, U_{n-1} = \frac{n(n-1)}{1}\cdot\frac{1.3.5\dots(2n-3)}{2^{n-1}}(-1)^{n-2}$$
$$\cdot(1+x+x^2)^{\frac{1}{2}-n+1}(1+2x)^{n-2},$$

$$\Gamma n\, U_{n-2} = \frac{n.(n-1)(n-2)(n-3)}{1.2}\cdot\frac{1.3.5\dots(2n-5)}{2^{n-2}}(-1)^{n-3}$$
$$(1+x+x^2)^{\frac{1}{2}-n+2}(1+2x)^{n-4}.$$

....

Hence, if for brevity we put $\frac{1+x+x^2}{(1+2x)^2} = z$, we have

$$f^n(x) = \frac{1.3.5\ldots(2n-1)}{2^n}(-1)^n . \frac{(1+2x)^n}{(1+x+x^2)^{n-\frac{1}{2}}}$$

$$\left\{1 - \frac{n-1}{1}\,\frac{2n}{2n-1}\,z + \frac{(n-2)(n-3)}{1.2}\,\frac{2n(2n-2)}{(2n-1)(2n-3)}\,z^2 + \&\text{c.}\right\},$$

as far as $z^{\frac{n}{2}}$ or $z^{\frac{n-1}{2}}$.

Example 2. Let $\phi(a + bx + cx^2) = \log(a + bx + x^2)$;

then $\phi^r(a + bx + cx^2) = (-1)^{r-1}\,\Gamma(r-1)\,(a + bx + x^2)^{-r}$;

$$\therefore\ U_r = \frac{\Gamma(r-1)\,.\,(-1)^{r-1}}{\Gamma(n-r)\,.\,\Gamma(2r-n)}(a + bx + x^2)^{-r}(b+2x)^{2r-n};$$

$$\therefore\ f^n(x) = \Gamma n\,(-1)^{n-1} . \frac{(b+2x)^n}{(a+bx+x^2)^n}\left\{\frac{1}{n} - z + \frac{z^2}{2n-4}\ldots\right\}.$$

The Cycloid.

Generation of the cycloid. If a circle *NPS* (fig. 73) roll on a straight line *AX*, a point *P* in the circumference of it will trace out a curve of the form *APBX*, which is called a *Cycloid.* It is a curve of some importance in Mechanics.

Equation to cycloid. We may find its equation as follows:

Let *A* be the point on the line *AX* with which *P* originally coincided, take $AM\,(=x)$ $MP\,(=y)$ to be the co-ordinates of *P*; draw *NC* from the center *C* to the point of contact *N* of the circle, join *CP* and draw *PO* parallel to *MN*, let a denote the radius *CP*, and θ the angle *NCP*; then, since in the rolling of the circle every point of the arc *PN* has coincided with every point of the line *AN*, it is evident that

$$AN = \text{arc } PN = a\theta;$$

therefore we have

$$\left.\begin{aligned} x &= AN - MN = a\theta - a\sin\theta \\ y &= CN - CO\ = a\ \ - a\cos\theta \end{aligned}\right\}\ldots\ldots(1);$$

$$\therefore\ y = a \operatorname{vers}\theta;\quad \text{and } \therefore\ \theta = \operatorname{vers}^{-1}\frac{y}{a};$$

$$\text{also, } \sin\theta = \sqrt{1 - \left(\frac{a-y}{a}\right)^2} = \frac{1}{a}\sqrt{2ay - y^2};$$

$$\therefore\ x = a \operatorname{vers}^{-1}\frac{y}{a} = \sqrt{2ay - y^2} \ldots\ldots\ldots (2),$$

which is the equation to the cycloid.

Equation transferred to vertex.

The point B at which $\theta = \pi$, which is called the vertex of the cycloid, is often taken for the origin, the line BY' parallel to AX for the axis of y, and the line BX' perpendicular to AX for the axis of x. We may easily find the equation to the cycloid referred to these axes by putting in the equations (1) $y' + AX'$ or $y' + \pi a$ for x, and $BX' - x'$ or $2a - x'$ for y, which gives us

$$y' + \pi a = a(\theta - \sin\theta),$$

$$2a - x' = a(1 - \cos\theta);$$

which, if we put $\pi + \theta'$ in the place of θ, become

$$\left.\begin{aligned} y' &= a(\theta' + \sin\theta') \\ x' &= a(1 - \cos\theta') \end{aligned}\right\} \ldots\ldots\ldots\ldots (3),$$

and therefore as before (suppressing the dashes),

$$y = a \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2} \ldots\ldots\ldots\ldots (4),$$

which is the equation to the cycloid referred to the vertex.

Tangent or normal to a cycloid.

To determine the tangent and normal at P.

From the equations (1) we find

$$dx = a(1 - \cos\theta)\,d\theta, \quad dy = a\sin\theta\, d\theta;$$

$$\therefore\ -\frac{dx}{dy} = -\frac{a(1 - \cos\theta)}{a\sin\theta}$$

$$= -\frac{PM}{MN} = \tan PNX';$$

therefore the equation to the normal at P is

$$y_1 - y = \tan PNX' \, (x_1 - x),$$

which is the equation to the chord PN.

Hence the chord PN *is the normal at* P, *and therefore the chord* PS, *which is perpendicular to* PN, *is the tangent at* P.

Radius of curvature.

To determine the index of curvature at P.

It follows from what has been just proved, that if ψ be the angle the tangent at P makes with the axis of x,

$$\psi = \frac{\pi}{2} - PSC = \frac{\pi}{2} - \frac{\theta}{2};$$

$$\therefore \rho = \frac{ds}{d\psi} = -\frac{2ds}{d\theta}.$$

$$\begin{aligned} \text{Now} \quad ds^2 &= dx^2 + dy^2 \\ &= a^2 \left\{(1 - \cos\theta)^2 + \sin^2\theta\right\} d\theta^2 \\ &= 2a^2 (1 - \cos\theta)\, d\theta^2 = 2ay\,d\theta^2; \end{aligned}$$

$$\therefore \rho^2 = 8ay,$$

which gives ρ.

Since $PN^2 = SN \cdot NO = 2ay$, this gives

$$\rho = 2\,PN;$$

therefore, if we produce PN till $NT = PN$, T is the center of curvature corresponding to P.

Evolute.

Let $AU (= \beta)$, $UT (= \alpha)$ be the co-ordinates of T, then

$$MN = NU = a \sin\theta,$$

$$\text{and} \quad UT = PM = a(1 - \cos\theta);$$

$$\therefore \beta = a(\theta + \sin\theta),$$

$$\alpha = a(1 - \cos\theta),$$

$$\text{and } \therefore\ \beta = a \operatorname{vers}^{-1}\frac{\alpha}{a} + \sqrt{2a\alpha - \alpha^2}.$$

Comparing this with equation (4) we see that the locus of T, i.e. the evolute is a cycloid $ATVXW$, as represented in the figure, being equal in magnitude to the original cycloid, but having its position altered, its vertex being at A, and its cusp being at V.

Length of any arc of a cycloid.

From the equation (1) we have

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{2a^2(1 - \cos\theta)\,d\theta^2}$$

$$= 2a \sin\frac{\theta}{2}\,d\theta = -4d\left(a\cos\frac{\theta}{2}\right);$$

$$\therefore\ d\left(s + 4a\cos\frac{\theta}{2}\right) = 0,$$

and $\therefore\ s + 4a\cos\frac{\theta}{2}$ must equal some constant, C suppose.

Now, assuming s to represent the arc AP it is evident that when $\theta = 0$, $s = 0$, therefore we have

$$0 + 4a = C, \quad \therefore\ C = a;$$

$$\therefore\ s = 4a\left(1 - \cos\frac{\theta}{2}\right),$$

which gives the length of the arc AP in terms of θ.

arc BP = 2 chord PS.

When P is at B, $\theta = \pi$, and $\therefore\ \cos\frac{\theta}{2} = 0$, hence the length of the arc AB is $4a$.

$$\text{Arc } BP = \text{arc } AB - \text{arc } AP$$

$$= 4a - 4a\left(1 - \cos\frac{\theta}{2}\right),$$

$$= 4a\cos\frac{\theta}{2}.$$

Now chord $PS = 2a\cos PSN = 2a\cos\frac{\theta}{2}$;

$$\therefore \text{ arc } BP = 2 \text{ chord } PS.$$

$s'^2 = 8ax'$. If we denote arc BP by s', then since

$$\text{chord } PS = \sqrt{SN \, . \, OS} = \sqrt{2ax'},$$

we have

$$s'^2 = 8ax',$$

which is the relation between the length of the arc and the abscissa measured from the vertex.

APPENDIX.

(A.)

(1) $$f(x) = \frac{3x^2 - 5}{x^2 + 1} = \frac{\phi(x)}{\psi(x)} \text{ suppose,}$$

$$\text{then } \phi'(x) = 6x, \quad \psi'(x) = 2x;$$

$$\therefore f'(x) = \frac{6x(x^2 + 1) - 2x(3x^2 - 5)}{(x^2 + 1)^2},$$

$$= \frac{16x}{(x^2 + 1)^2}.$$

(2) $$f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}, \quad f'(x) = \frac{2(x^2 + 1)}{(x^2 + x + 1)^2}.$$

(3) $$f(x) = \frac{x^2 + 1}{x^2 - 1} + \frac{x - 1}{x + 1} + x,$$

$$f'(x) = \frac{-4x}{(x^2 - 1)^2} + \frac{2}{(x + 1)^2} + 1 = \frac{x^4 - 8x + 3}{(x^2 - 1)^2}.$$

(4) $$f(x) = \frac{x^3}{x^2 + 1} + \frac{x^2}{x + 1} + x.$$

(5) $$f(x) = \frac{(x^2 - 1)(x + 2)}{x - 3} = \frac{\phi(x)}{x - 3} \text{ suppose,}$$

$$\text{then } f'(x) = \frac{\phi'(x)(x - 3) - \phi(x)}{(x - 3)^2},$$

$$\phi'(x) = 2x(x + 2) + (x^2 - 1) = 3x^2 + 4x - 1;$$

$$\therefore f'(x) = \frac{(3x^2+4x-1)(x-3)-(x^2-1)(x+2)}{(x-3)^2} = \frac{2x^3-7x^2-12x+5}{(x-3)^2}.$$

(6) $$f(x) = \frac{(x-2)(x-3)}{(x-1)(x-4)}.$$

(7) $$f(x) = 2x^{\frac{3}{2}} - \frac{2}{x^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}} - 1}{x^{\frac{1}{2}} + 1},$$

$$f'(x) = 3x^{\frac{1}{2}} + x^{-\frac{3}{2}} + \frac{x^{-\frac{1}{2}}}{x + 1 + 2x^{\frac{1}{2}}}.$$

(8) $$f(x) = \frac{x^n - a^n}{x^n + a^n}, \quad f'(x) = \frac{2nx^{n-1}a^n}{(x^n + a^n)^2}.$$

(B.)

(1) $$f(x) = \sqrt{z} = \phi(z), \quad z = 1 + e^x = \psi(x),$$

then $$f'(x) = \phi'(z)\,\psi'(x) = \frac{e^x}{2\sqrt{z}}.$$

(2) $$f(x) = \tan z = \phi(z), \quad z = m\tan^{-1}x = \psi(x),$$

then $$f'(x) = \phi'(z)\,\psi'(x) = \frac{1}{\cos^2 z}\cdot\frac{m}{1 + x^2}.$$

(3) $$f(x) = \log z = \phi(z), \quad z = \sin x + \tan x = \psi(x),$$

then $$f'(x) = \phi'(z)\,\psi'(x) = \frac{1}{z}\left(\cos x + \frac{1}{\cos^2 x}\right).$$

(4) $$f(x) = \cos^{-1} z = \phi(z), \quad z = \epsilon^x \sin x = \psi(x),$$

then $$f'(x) = \phi'(z)\,\psi'(x) = -\frac{1}{\sqrt{1 - z^2}}\,.\,\epsilon^x(\sin x + \cos x).$$

(C.)

(1) $$y = \frac{x}{\sqrt{1 - x^2}} = x(1 - x^2)^{-\frac{1}{2}},$$

$$dy = dx\,.\,(1 - x^2)^{-\frac{1}{2}} + x\,d\,(1 - x^2)^{-\frac{1}{2}},$$

$$d(1-x^2)^{-\frac{1}{2}} = -\tfrac{1}{2}(1-x^2)^{-\frac{3}{2}}\,d(1-x^2) = x(1-x^2)^{-\frac{3}{2}}\,dx\,;$$

$$\therefore\ \frac{dy}{dx} = (1-x^2)^{-\frac{1}{2}} + x^2(1-x^2)^{-\frac{3}{2}} = \frac{1}{(1-x^2)^{\frac{3}{2}}}\,;$$

Or thus:

$$\log y = \log x - \tfrac{1}{2}\log(1-x^2)\,;$$

$$\therefore\ \frac{dy}{y} = \frac{dx}{x} - \frac{1}{2}\,\frac{-2x\,dx}{1-x^2} = \frac{1}{x}\cdot\frac{dx}{1-x^2}\,;$$

$$\therefore\ \frac{dy}{dx} = \frac{y}{x}\,\frac{1}{1-x^2} = \frac{1}{(1-x^2)^{\frac{3}{2}}}.$$

(2) $$y = \log(x\sin x + \cos^2 x)$$

$$dy = \frac{d(x\sin x + \cos^2 x)}{x\sin x + \cos^2 x}$$

$$= \frac{dx\,.\sin x + x\cos x\,dx + 2\cos x\,d\cos x}{x\sin x + \cos^2 x}\,;$$

$$\therefore\ \frac{dy}{dx} = \frac{\sin x(1-2\cos x) + x\cos x}{x\sin x + \cos^2 x}.$$

(3) $$y = (\tan x)^{\sec x}\,;$$

$$\therefore\ \log y = \frac{1}{\cos x}\log\tan x,$$

$$\frac{dy}{y} = -\frac{d\cos x}{\cos^2 x}\log\tan x + \frac{1}{\cos x}\,\frac{d\tan x}{\tan x}\,;$$

$$\therefore\ \frac{dy}{dx} = y\left\{\frac{a\sin x}{\cos^2 x}\log\tan x + \frac{1}{\sin x\cos^2 x}\right\}.$$

(4) $$y = \log\frac{x}{\sqrt{a^2+x^2}} = \log x - \tfrac{1}{2}\log(a^2+x^2)\,;$$

$$\therefore\ dy = \frac{dx}{x} - \frac{1}{2}\,\frac{2x\,dx}{a^2+x^2}\,;$$

$$\therefore\ \frac{dy}{dx} = \frac{a^2}{x(a^2+x^2)}.$$

(5) $$y = \tan^{-1}\left(\frac{\epsilon^x \sin x}{1 + \epsilon^x \cos x}\right);$$

then $\log \tan y = x + \log \sin x - \log(1 + \epsilon^x \cos x)$;

$$\therefore \frac{d \tan y}{\tan y} = \frac{dy}{\cos y \sin y} = dx + \frac{\cos x\, dx}{\sin x} - \frac{\epsilon^x (\cos x - \sin x)\, dx}{1 + \epsilon^x \cos x};$$

$$\therefore \frac{dy}{dx} = \cos y \sin y \,.\left\{\frac{\sin x + \cos x + \epsilon^x}{(1 + \epsilon^x \cos x) \sin x}\right\}.$$

$$\text{Now } \sin y = \frac{\tan y}{\sqrt{1 + \tan^2 y}} = \frac{\epsilon^x \sin x}{\sqrt{1 + 2\epsilon^x \cos x + \epsilon^{2x}}},$$

$$\cos y = \frac{1}{\sqrt{1 + \tan^2 y}} = \frac{(1 + \epsilon^x \cos x)}{\sqrt{1 + 2\epsilon^x \cos x + \epsilon^{2x}}};$$

$$\therefore \frac{dy}{dx} = \frac{\epsilon^x (\sin x + \cos x + \epsilon^x)}{1 + 2\epsilon^x \cos x + \epsilon^{2x}}.$$

Or thus:

$$dy = \frac{dz}{1 + z^2}, \text{ where } z = \frac{\epsilon^x \sin x}{1 + \epsilon^x \cos x},$$

and therefore,

$$dz = \frac{d(\epsilon^x \sin x) . (1 + \epsilon^x \cos x) - \epsilon^x \sin x\, d(\epsilon^x \cos x)}{(1 + \epsilon^x \cos x)^2}$$

$$= \frac{\epsilon^x (\sin x + \cos x)(1 + \epsilon^x \cos x) - \epsilon^x \sin x\, \epsilon^x (\cos x - \sin x)}{(1 + \epsilon^x \cos x)^2} dx;$$

$$\therefore \frac{dy}{dx} = \frac{1}{1 + z^2} \frac{\epsilon^x (\sin x + \cos x + \epsilon^x)}{(1 + \epsilon^x \cos x)^2}$$

$$= \frac{\epsilon^x (\sin x + \cos x + \epsilon^x)}{1 + 2\epsilon^x \cos x + \epsilon^{2x}}.$$

(6) $$y = \log\left(\frac{x - \sqrt{1 - x^2}}{x + \sqrt{1 - x^2}}\right)$$

$$= \log(x - \sqrt{1 - x^2}) - \log(x + \sqrt{1 - x^2});$$

$$\therefore\ dy = \frac{dx + \dfrac{x\,dx}{\sqrt{1-x^2}}}{x - \sqrt{1-x^2}} - \frac{dx - \dfrac{x\,dx}{\sqrt{1-x^2}}}{x + \sqrt{1-x^2}};$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}} + \frac{x-\sqrt{1-x^2}}{x+\sqrt{1-x^2}}\right)$$

$$= \frac{1}{\sqrt{1-x^2}}\left(\frac{1}{2x^2-1}\right).$$

(7) $$y = \cfrac{x^m}{1 - \cfrac{x^m}{1 - \cfrac{x^m}{1 \ldots}}}\ \ldots\ldots$$

then $y = \dfrac{x^m}{1-y}$; and $\therefore\ y - y^2 = x^m$;

$$\therefore\ dy - 2y\,dy = mx^{m-1}dx,$$

$$\frac{dy}{dx} = \frac{mx^{m-1}}{1-2y} = \cfrac{mx^{m-1}}{1 - \cfrac{2x^m}{1 - \cfrac{x^m}{1 - \cfrac{x^m}{\ldots\ldots}}}}\ \ldots\ldots$$

(8) $$y = \log\,(x + \sec x + \sin^{-1} x),$$

$$dy = \frac{1 + d\sec x + d\sin^{-1} x}{x + \sec x + \sin^{-1} x}.$$

(9) $$y = \tan^{-1}\frac{1}{\sqrt{1-x^2}};$$

$$\therefore\ \log\tan y = \log x - \tfrac{1}{2}\log\,(1-x^2),$$

$$\frac{dy}{\sin y\cos y} = \frac{dx}{x} + \frac{x\,dx}{1-x^2}.$$

(10) $$y = \log \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$$

$$= \tfrac{1}{2}\log(1+\sin\theta) - \tfrac{1}{2}\log(1-\sin\theta),$$

$$dy = \frac{1}{2}\frac{\cos\theta d\theta}{1+\sin\theta} + \frac{1}{2}\frac{\cos\theta d\theta}{1-\sin\theta},$$

$$\frac{dy}{dx} = \frac{\cos\theta}{2}\,\frac{2}{\cos^2\theta} = \frac{1}{\cos\theta}.$$

(11) $$y = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

$$dy = \frac{d\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)},$$

and $$\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right) = \tfrac{1}{2}\sin\left(\frac{\pi}{2}+x\right);$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos x}.$$

(12) $$y = \frac{1}{\sqrt{a^2-b^2}}\cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right),$$

$$\log\cos ny = \log(b+a\cos x) - \log(a+b\cos x) \quad \{n=\sqrt{a^2-b^2}\};$$

$$\therefore \frac{d\cos ny}{\cos ny} \text{ or } -\frac{u\sin ny dy}{\cos ny} = \frac{-a\sin x dx}{b+a\cos x} + \frac{b\sin x dx}{a+b\cos x}.$$

$$\therefore \frac{dy}{dx} = \frac{1}{n}\cot ny\sin x\left(\frac{a}{b+a\cos x} - \frac{b}{a+b\cos x}\right)$$

$$= \cot ny\sin x\,\frac{n}{(b+a\cos x)(a+b\cos x)}.$$

Now since $\cos ny = \dfrac{b+a\cos x}{a+b\cos x}$;

$$\therefore \ \cot ny = \frac{b + a\cos x}{\sqrt{(a + b\cos x)^2 - (b + a\cos x)^2}} = \frac{b + a\cos x}{n \sin x};$$

$$\therefore \ \frac{dy}{dx} = \frac{1}{a + b\cos x}.$$

(D.)

(1) $y = \epsilon^{mx}$, $\dfrac{d^n y}{dx^n} = m^n \epsilon^{mx}$.

(2) $y = x^m$, $\dfrac{d^n y}{dx^n} = m\,.\,(m-1)(\ldots)(m-n+1)\,x^{m-n}$.

(3) $y = \log x$, $\dfrac{d^n y}{dx^n} = (-)^{n-1}.\,\Gamma(n-1)\,.\,x^{-n}$.

(4) $y = \sin(x+a)$, $\dfrac{d^n y}{dx^n} = \sin\left(x + a + \dfrac{n\pi}{2}\right)$.

(5) $y = \tan x$,

$$\frac{d^2 y}{dx^2} = \frac{2\sin x}{\cos^3 x},$$

$$\frac{d^3 y}{dx^3} = \frac{2}{\cos^2 x} + \frac{6\sin^2 x}{\cos^4 x}$$

$$= -\frac{4}{\cos^2 x} + \frac{6}{\cos^4 x};$$

&c. &c.

Hence assume that in general

$$\frac{d^{2n-1} y}{dx^{2n-1}} = \frac{A_n}{\cos^2 x} + \frac{B_n}{\cos^4 x} + \frac{C_n}{\cos^6 x} \ldots\ldots$$

and we find

$$\frac{d^{2n} y}{dx^{2n}} = \frac{2A_n \sin x}{\cos^3 x} + \frac{4B_n \sin x}{\cos^5 x} + \frac{6C_n \sin x}{\cos^7 x} \ldots$$

$$\frac{d^{2n+1}y}{dx^{m+1}} = \frac{2A_n}{\cos^2 x} + \frac{4B_n}{\cos^4 x} + \frac{6C_n}{\cos^6 x} \dots$$

$$+ \frac{2 \, . \, 3A_n \sin^2 x}{\cos^4 x} + \frac{4 \, . \, 5B_n \sin^2 x}{\cos^6 x} + \frac{6 \, . \, 7C_n \sin^2 x}{\cos^8 x} \dots$$

$$= \frac{-2^2 A_n}{\cos^2 x} + \frac{2 \, . \, 3A_n - 4^2 B_n}{\cos^4 x} + \frac{4 \, . \, 5B_n - 6^2 C_n}{\cos^8 x} \dots$$

Hence we have

$$A_{n+1} = -2^2 A_n, \quad B_{n+1} = 2 \, . \, 3A_n - 4^2 B_n,$$

$$C_{n+1} = 4 \, . \, 5B_n - 6^2 C_n \dots \text{ \&c. \quad \&c.}$$

From these equations we may find by successive substitution the quantities A_n, B_n, &c.

The general expression for $\frac{d^n y}{dx^n}$ is very complicated in this case.

(6) $y = x\epsilon^x$;

$$\therefore \frac{dy}{dx} = (x+1)\epsilon^x,$$

$$\frac{d^2 y}{dx^2} = (x+2)\epsilon^x,$$

$$\dots\dots\dots\dots\dots$$

$$\frac{d^n y}{dx^n} = (x+n)\epsilon^x.$$

(7) $y = x^m \epsilon^x$,

$$\frac{dy}{dx} = (x^m + m x^{m-1})\epsilon^x,$$

$$\frac{d^2 y}{dx^2} = \{x^m + 2m x^{m-1} + m \, . \, (m-1) \, . \, x^{m-2}\}\epsilon^x.$$

By Leibnitz' Theorem, see Chap. XIX., we obtain immediately, putting $u = \epsilon^x$, $v = x^m$,

$$\frac{d^n y}{dx^n} = \epsilon^x \left\{x^m + \frac{n}{1} m x^{m-1} + \frac{n.(n-1)}{\Gamma 2} m(m-1).x^{m-2}\ldots\right\}.$$

(8) $y = \epsilon^x \sin x$.

We find similarly by Leibnitz' Theorem,

$$\frac{d^n y}{dx^n} = \epsilon^x \left\{\sin x + \frac{n}{1}\cos x - \frac{n(n-1)}{\Gamma 2}\sin x - \frac{n(n-1)(n-2)}{\Gamma 3}\cos x + \ldots\right\}$$

$$= \epsilon^x \left[\left\{1 - \frac{n.(n-1)}{\Gamma 2} + \frac{n(n-1).(n-2)(n-3)}{\Gamma 4}\ldots\right\}\sin x + \left\{\frac{n}{1} - \frac{n(n-1)(n-2)}{\Gamma 3}\ldots\right\}\cos x\right]$$

$$= \frac{\epsilon^x}{2}\left[\left\{(1+\sqrt{-1})^n + (1-\sqrt{-1})^n\right\}\sin x + \frac{1}{\sqrt{-1}}\left\{(1+\sqrt{-1})^n - (1-\sqrt{-1})^n\right\}\cos x\right].$$

Now $1 \pm \sqrt{-1} = \sqrt{2}\left(\cos\frac{\pi}{4} \pm \sin\frac{\pi}{4}\sqrt{-1}\right)$;

$$\therefore\ (1 \pm \sqrt{-1})^n = 2^{\frac{n}{2}}\left(\cos\frac{m\pi}{4} \pm \sin\frac{n\pi}{4}\sqrt{-1}\right);$$

$$\therefore\ \frac{d^n y}{dx^n} = 2^{\frac{n}{2}}\epsilon^x\left(\sin x\cos\frac{n\pi}{4} + \cos x\sin\frac{n\pi}{4}\right)$$

$$= 2^{\frac{n}{2}}\epsilon^x \sin\left(x + \frac{n\pi}{4}\right).$$

Or thus more generally and simply.

Let $y = \epsilon^{mx}\sin(x + a)$,

$$\text{then } \frac{dy}{dx} = \epsilon^{mx}\left\{\sin(x+\alpha) + m\cos(x+\alpha)\right\}$$

$$= \frac{\epsilon^{mx}}{\cos\beta}\sin(x+\alpha+\beta),$$

if we put $m = \tan\beta$;

∴ in a similar manner we find,

$$\frac{d^2y}{dx^2} = \frac{\epsilon^{mx}}{\cos^2\beta}\sin(x+\alpha+2\beta);$$

and ∴ in general $\dfrac{d^ny}{dx^n} = \dfrac{\epsilon^{mx}}{\cos^n\beta}\sin(x+\alpha+n\beta)$;

(9) $y = \tan^{-1}x$,

then $\dfrac{dy}{dx} = \cos^2 y$,

$$\frac{d^2y}{dx^2} = -2\sin y\cos y\frac{dy}{dx} = -\sin 2y\cos^2 y$$

$$= \cos\left(2y + \frac{\pi}{2}\right)\cos^2 y,$$

$$\frac{d^3y}{dx^3} = -2\left\{\cos\left(2y+\frac{\pi}{2}\right)\sin y + \sin\left(2y+\frac{\pi}{2}\right)\cos y\right\}\cos y\frac{dy}{dx}$$

$$= 2\cos(3y+\pi)\cos^2 y$$

……………………………

Now if $u = A\cos(ny+\beta)\cos^n y$,

$$\frac{du}{dx} = -nA\left\{\cos(ny+\beta)\sin y + \sin(ny+\beta)\cos y\right\}\cos^{n-1}y\frac{dy}{dx}$$

$$= nA\cos\left\{(n+1)y + \beta + \frac{\pi}{2}\right\}\cos^{n+1}y;$$

from which it is clear, that if we assume

$$\frac{d^ny}{dx^n} = \Gamma(n-1)\cos\left\{ny + (n-1)\cdot\frac{\pi}{2}\right\}\cos^n y,$$

the same law will hold when $n+1$ is put for n, but we have found this law to be true for 2 and 3, therefore it is true in general; and it gives us the n^{th} differential coefficient of $\tan^{-1} x$.

(E.)

(1) Having given $u = \dfrac{ds^3}{d^2y\,dx}$,

where dx is constant, and $ds^2 = dx^2 + dy^2$; to find what u becomes if s be made the independant variable.

We have $\quad d^2y = dx\,d\left(\dfrac{dy}{dx}\right)$;

$$\therefore\ d^2y\,dx = d^2y\,dx - dy\,d^2x \ldots (1).$$

Now since $ds^2 = dx^2 + dy^2$, and ds is constant, we have

$$0 = dx\,d^2x + dy\,d^2y \ldots (2),$$

$(1)^2 + (2)^2$ gives, observing that $dx^2 + dy^2 = ds^2$,

$$(d^2y\,dx)^2 = (d^2y)^2 + (d^2x)^2.$$

Hence $u = \dfrac{ds^3}{\sqrt{(d^2y)^2 + (d^2x)^2}}$,

in which expression s is the independant variable.

(2) Given $u = \dfrac{ds^3}{d^2y\,dx}$,

where dx is constant, and $x = a\cos nt$, $y = b\sin nt$; to make t the independant variable,

$$dx = -na\sin nt\,dt, \quad dy = nb\cos nt\,dt,$$

$$ds^2 = n^2(a^2\sin^2 nt + b^2\cos^2 nt)\,dt^2,$$

$$\therefore\ \frac{dy}{dx} = -\frac{b}{a}\cot nt,$$

$$dx d^2y = dx^2 d\left(\frac{dy}{dx}\right) = \frac{b}{a}\,\frac{n\,dt}{\sin^2 nt}\,n^2a^2\sin^2 nt\,dt^2$$

$$= n^3ab\,dt^3;$$

$$\therefore\ u = \frac{(a^2\sin^2 nt + b^2\cos^2 nt)^{\frac{3}{2}}}{ab}.$$

(F.)

(1) $y = e^v \tan u + \log(u+v)$,

$$d^2y = \left(e^v\tan u + \frac{1}{u+v}\right)d^2v + \left(\frac{e^v}{\cos^2 u} + \frac{1}{u+v}\right)d^2u$$

$$+ \left\{e^v\tan u - \frac{1}{(u+v)^2}\right\}dv^2 + 2\left\{\frac{e^v}{\cos^2 u} - \frac{1}{(u+v)^2}\right\}du\,dv$$

$$+ \left\{\frac{2e^v\sin u}{\cos^3 u} - \frac{1}{(u+v)^2}\right\}du^2.$$

(2) $y = u^{u+v} + v^{u+v}$;

put $u + v = z$, then $y = u^z + v^z$;

$$\therefore\ dy = (u^z\log u + v^z\log v)\,dz + zu^{z-1}du + zv^{z-1}dv,$$

and $\therefore$ since $dz = du + dv$,

$$dy = \{u^{u+v}\log u + v^{u+v}\log v + (u+v)\,u^{u+v-1}\}\,du$$

$$+ \{u^{u+v}\log u + v^{u+v}\log v + (u+v)\,v^{u+v-1}\}\,dv.$$

(3) $y\sin x + x\sin y - xy = 0$,

$$y\cos x + \sin y - y + (\sin x + x\cos y - x)\frac{dy}{dx} = 0$$

$$-y\sin x + 2(\cos x + \cos y - 1)\frac{dy}{dx} - x\sin y\left(\frac{dy}{dx}\right)^2$$

$$+ (\sin x + x\cos y - x)\frac{d^2y}{dx^2} = 0,$$

&c. ... &c.,

whence $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. may be determined.

(G.)

(1) Let $y = \sin^{-1} x$;

$$\text{then } \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} = 1 + \tfrac{1}{2} x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4 \ldots\ldots$$

$$\frac{1 \cdot 3 \ldots 2n - 1}{2 \cdot 4 \ldots 2n} x^n + \&\text{c.};$$

$$\therefore \frac{d^{n+1} y}{dx^{n+1}} = \frac{1 \cdot 3 \ldots 2n - 1}{2 \cdot 4 \ldots 2n} \cdot \Gamma n + \text{powers of } x$$

$$= \frac{1 \cdot 3 \ldots 2n - 1}{2 \cdot 4 \ldots 2n} \Gamma n, \text{ when } x = 0.$$

Hence the general term of the expansion of y in powers of x is

$$\frac{1 \cdot 3 \ldots 2n - 1}{2 \cdot 4 \ldots 2n} \cdot \frac{x^{n+1}}{n + 1};$$

and $\therefore$ since $\sin^{-1} x = m\pi$ when $x = 0$, we have

$$\sin^{-1} x = m\pi + \frac{x}{1} + \tfrac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} \ldots \&\text{c.}$$

(2) We may often obtain developements of this kind more simply in the following manner.

$$\text{Assume } \quad y = A_0 + A_1 x + A_2 x^2 \ldots \&\text{c.},$$

$$\text{and } \therefore \frac{dy}{dx} = A_1 + 2A_2 x + 3A_3 x^2 \ldots \&\text{c.}$$

Comparing this with the above value of $\frac{dy}{dx}$ we find immediately

$$A_1 = 1, \; A_2 = 0, \; A_3 = \tfrac{1}{3} \tfrac{1}{2}, \; A_4 = 0, \; A_5 = \tfrac{1}{5} \frac{1 \cdot 3}{2 \cdot 4} \ldots \&\text{c.}$$

Also $A_0 =$ the value of y when $x = 0$, which is $m\pi$;

$$\text{hence } \quad y = m\pi + \frac{x}{1} + \tfrac{1}{2} \frac{x^3}{3} + \&\text{c.} \ldots \text{as before.}$$

(3) $$y = \sin x \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Assume $$y = A_0 + A_1 x + A_2 x^2 \ldots\ldots\ldots\ldots (2),$$

then from (1) we have

$$\frac{d^2y}{dx^2} = -\sin x = -A_0 - A_1 x - A_2 x^2 \ldots\ldots(3),$$

and from (2) we have

$$\frac{d^2y}{dx^2} = 1 \,.\, 2\, A_2 + 2 \,.\, 3\, A_3 x + 3 \,.\, 4\, A_4 x^2 \ldots\ldots(4).$$

$$\text{Now } A_0 = y \text{ when } x = 0 ; \;\therefore\; A_0 = 0 ;$$

$$\text{and } A_1 = \frac{dy}{dx} = \cos x \text{ when } x = 0 ; \;\therefore\; A_1 = 1 ;$$

and therefore, comparing (3) and (4), we have

$$A_2 = 0, \;\; A_3 = -\frac{1}{\Gamma 3}, \;\; A_4 = 0, \;\; A_5 = -\frac{A_3}{4 . 5} = \frac{1}{\Gamma 5} \ldots \&c. ;$$

$$\therefore\; \sin x = \frac{x}{\Gamma 1} - \frac{x^3}{\Gamma 3} + \frac{x^5}{\Gamma 5} \ldots \&c.$$

(4) If we differentiate the result just obtained, we find

$$\cos x = 1 - \frac{x^2}{\Gamma 2} + \frac{x^4}{\Gamma 4} \ldots \&c.$$

(5) If $\tan y = a \tan x$, to expand y in powers of x.

(6) If $\sin y = a \sin (y + x)$, to expand y in powers of x.

(H.)

(1) To expand $\cot x$ in powers of x.

$\cot x = \infty$ when $x = 0$; therefore there are negative powers in the developement, and Taylor's series fails. We may proceed as follows:

$$\cot x = \frac{\cos x}{\sin x} = \frac{1 - \frac{x^2}{\Gamma 2} + \frac{x^4}{\Gamma 4} \dots}{\frac{x}{\Gamma 1} - \frac{x^3}{\Gamma 3} \dots \&c.}$$

$$= \frac{1}{x} \cdot \frac{1 - \frac{x^2}{\Gamma 2} \dots \&c.}{1 - \frac{x^2}{\Gamma 3} \dots \&c.};$$

assume this to be equal to

$$\frac{1}{x}(A + Bx + Cx^2 \dots);$$

then we have

$$(A + Bx + Cx^2 \dots)(1 - \frac{x^2}{\Gamma 3} \dots) = 1 - \frac{x^2}{\Gamma 2} \dots$$

$$\text{or } A + Bx + \left(C - \frac{A}{\Gamma 3}\right) x^2 \dots = 1 - \frac{x^2}{\Gamma 2} \dots$$

$$\therefore A = 1, \quad B = 0, \quad C = \frac{1}{\Gamma 3} - \frac{1}{\Gamma 2} = -\tfrac{1}{3}, \text{ \&c.}$$

$$\therefore \cot x = \frac{1}{x}(1 - \frac{x^2}{3} \dots \&c.).$$

(2) Let $y = (e^x - 1)^{\frac{1}{3}}$.

Here $\frac{dy}{dx} = \frac{1}{3} e^x (e^x - 1)^{-\frac{2}{3}} = \infty$ when $x = 0$:

therefore fractional powers occur in the developement, and Taylor's series fails. We may proceed as follows:

$$e^x - 1 = \frac{x}{\Gamma 1} + \frac{x^2}{\Gamma 2} \dots$$

$$\therefore y = x^{\frac{1}{3}}(1 + \frac{x}{\Gamma 2} + \frac{x^2}{\Gamma 3} \dots)^{\frac{1}{3}}$$

$$= x^{\frac{1}{3}}(1 + \tfrac{1}{3}\frac{x}{\Gamma 2} \dots \&c.),$$

by the binomial theorem.

(3) $$y = 1 + x \log (e + \sqrt{x}).$$

$$\text{Here } \frac{d^2y}{dx^2} = \infty, \quad \text{and } y = 1 + x + \frac{x^{\frac{3}{2}}}{e} \ldots \&\text{c.}$$

(I.)

The following remark is of some importance, as it will often enable us to simplify the process explained in (145).

Suppose that we wish to expand y in powers of x, having given a relation between y and x; then if $y = a$ when $x = 0$, we assume $y = a + ux^m$, as is explained in (145).

Now suppose that the result of this substitution comes out in the form

$$U_1x^{r_1} + U_2x^{r_2} + U_3x^{r_3} + \&\text{c.} = 0,$$

and suppose that we can see by inspection that one of the indices r_2 must be greater than another r_1 whatever value m may have, (m of course is always positive); then we may *immediately reject* the term $U_2x^{r_2}$, since it cannot be one of the terms containing the *lowest* power of x, and therefore cannot affect the result, as is evident from the example in (145).

(1) Let $y^4 - x^2y^2 + ax^2y - a^2x^2 = 0$;

here $y = 0$ when $x = 0$;

assume $\therefore$ $y = ux^m$; then

$$u^4x^{4m} - u^2x^{2m+2} + aux^{m+2} - a^2x^2 = 0.$$

Now here $2m + 2$ must be $> m + 2$ whatever m be, and $m + 2$ must be > 2; hence we may immediately reject the terms u^2x^{2m+2} and aux^{m+2}; and $\therefore$ we have simply

$$u^4x^{4m} - a^2x^2 = 0,$$

which gives $4m = 2$, and $\therefore$ $m = \frac{1}{2}$, and $\therefore$ $u = \pm\sqrt{a}$,

$$\text{and } \therefore y = \pm\sqrt{a} \,.\, x^{\frac{1}{2}} + R.$$

Of course we must not reject these terms if we proceed to find the second and higher terms of the expansion.

(2) $$y^4 - x^2y^2 + axy^2 - a^2x^3 = 0;$$

$$\therefore\ u^4x^{4m} - u^2x^{2m+2} + au^2x^{2m+1} - a^2x^3;$$

or since $2m + 2$ is $> 2m + 1$,

$$u^4x^{4m} - au^2x^{2m+1} - a^2x^3.$$

Suppose that m is > 1, then $4m$ is $> 2m + 1$, and $2m + 1$ is > 3, which cannot be; therefore m is < 1; $\therefore$ 3 is $> 2m + 1$; $\therefore$ a^2x^3 is to be rejected, and $\therefore$ we have

$$4m = 2m + 1, \text{ and } \therefore\ m = \tfrac{1}{2}, \text{ and } \therefore\ u = \pm\sqrt{a};$$

$$\therefore\ y = \pm\sqrt{a}\,.\,x^{\frac{1}{2}} + R.$$

(J.)

(1) Let $y = x\log x$; to find the limiting value of y when x approaches zero.

We have $y = \dfrac{x}{\dfrac{1}{\log x}} = \dfrac{0}{0}$ when $x = 0$;

$\therefore$ by Lemma XX., y and $\dfrac{dx}{d\dfrac{1}{\log x}}$

have the same limiting value when x approaches 0.

Now $$\frac{dx}{d\dfrac{1}{\log x}} = -\,x(\log x)^2 = -\frac{y^2}{x};$$

$\therefore$ y and $-\dfrac{y^2}{x}$ have the same limiting value when x approaches 0;

therefore, multiplying each of these quantities by $\frac{x}{y}$, x and $-y$ have the same limiting value when x approaches zero, and therefore 0 is the limiting value of y.

Here we assume, that if any functions U and V have the same limiting value when x approaches a, the same is true of WU and WV, W being any other function; which is evidently true, since if A be the limiting value of U and V, and B that of W, then by Lemma VIII, BA and BA are the limiting values of WU and WV.

(2) Let $y = xe^{-x}$; to find the limiting value of y when x approaches ∞.

$$y = \frac{e^{-x}}{\frac{1}{x}} = \frac{0}{0} \text{ when } x = \infty ;$$

$\therefore$ y and $\dfrac{de^{-x}}{d\frac{1}{x}}$ have the same limiting value when x approaches ∞.

Now $\dfrac{de^{-x}}{d\frac{1}{x}} = x^2 e^{-x} = \dfrac{y^2}{e^{-x}}$;

$\therefore$ y and e^{-x} have the same limiting value when x approaches ∞, therefore the limiting value of y is 0.

(3) Let $y = (\sin x)^{\tan x}$; to find the limiting value of y when x approaches 0.

$$\log y = \tan x \log \sin x = \frac{1}{\cos x} z \log z,$$

if we put $\sin x = z$; hence, since z approaches zero when x does, and since the limiting value of $\cos x$ is 1, and that of $z \log z$ zero, the limiting value of $\log y$ is zero, and therefore that of y is unity.

(K.)

(1) To inscribe the greatest rectangle in a semi-circle.

Let $CM = x$, $MP = y$ (fig. 33), then $2xy$ is the area of the inscribed rectangle; hence we have

$$d(2xy) = 0; \quad \text{and } \therefore \; d(x^2y^2) = 0;$$

$$\text{or} \quad d\left\{x^2(a^2 - x^2)\right\} = 0;$$

$$\therefore \; 2a^2x - 4x^3 = 0,$$

which is satisfied by $x = \dfrac{a}{\sqrt{2}}$ and $x = 0$. $x = \dfrac{a}{\sqrt{2}}$ evidently gives a maximum.

(2) To inscribe the greatest semi-ellipse in an isosceles triangle. (fig 34.)

Let $CM = x$, $MP = y$, then if $CE = h$, $CF = k$, we have

$$h = \frac{a^2}{x}, \quad k = \frac{b^2}{y};$$

$$\therefore \; \frac{a^2}{h^2} + \frac{b^2}{k^2} = 1 \; \ldots\ldots \; (1).$$

Also πab is to be a maximum;

$$\text{and } \therefore \; d(ab) = 0;$$

$$\therefore \; bda + adb;$$

$$\text{and by (1),} \qquad \frac{ada}{h^2} + \frac{bdb}{k^2} = 0;$$

$$\therefore \; \frac{a}{bk^2} = \frac{b}{ak^2},$$

which, along with (1), gives a and b.

(3) To find the greatest triangle that can be inscribed in a given circle.

ABC (fig. 35) is a maximum, supposing AB invariable, if $AC = BC$, as may be easily shewn. Therefore *any* two sides of the maximum triangle must be equal; therefore it must be an equilateral triangle.

(4) To find the greatest quadrilateral figure that can be circumscribed about a circle.

We may prove similarly that it must be a square.

(5) To find the greatest triangle which has a given perimeter.

(6) To find the greatest polygon that has a given perimeter.

Suppose all the sides but three invariable.

(L.)

(1) The normal to any curve is in general the greatest or least line which can be drawn to a given point to it.

Let xy be any point on the curve, $x'y'$ any other point, and R the distance between them; then

$$R^2 = (x - x')^2 + (y - y')^2,$$

$$R\,dR = (x - x')\,dx + (y - y')\,dy,$$

but $dR = 0$ if R be a maximum or minimum; and therefore

$$x' - x + (y' - y)\frac{dy}{dx} = 0,$$

which shews that $x'y'$ is a point of the normal at xy.

(2) To draw a tangent to a given curve, cutting off the greatest or least area from the space included between the co-ordinate axes.

The equation to the tangent being

$$(y' - y)\,dx - (x' - x)\,dy = 0,$$

the portions it cuts off the axes are

$$\beta = \frac{y\,dx - x\,dy}{dx}, \quad \text{and } \alpha = \frac{x\,dy - y\,dx}{dy};$$

$$\text{and } \frac{\alpha\beta}{2} \text{ or } -\frac{(ydx - xdy)^2}{2dxdy}$$

is to be a maximum.

Therefore, considering dx as constant, we have

$$d\left\{\frac{(ydx - xdy)^2}{dy}\right\} = 0,$$

$$\text{or } \frac{-2(ydx - xdy)\,xd^2ydy - (ydx - xdy)^2 d^2y}{dy^2} = 0,$$

$$\text{which gives } 2x = -\frac{ydx - xdy}{dy} = -\alpha;$$

$$\therefore \; x = \frac{\alpha}{2}:$$

this equation shews that the portion of the tangent intercepted between the co-ordinate axes is bisected at the point of contact when it cuts off a maximum or minimum area.

(3) In the ellipse, if we assume $x = a\cos t$, then we have $y = b\sin t$, in virtue of the equation

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}.$$

Hence we may always suppose in the ellipse, that

$$x = a\cos t, \quad \text{and} \quad y = b\sin t.$$

If we describe a circle on the major-axis as diameter, and if we produce the ordinate MP to meet this circle in Q, then it is evident that $\angle QCA = t$, for

$$\cos QCA = \frac{CM}{MQ} = \frac{x}{a} = \cos t.$$

This transformation is often useful.

Differentiating, we have

$$dx = -a\sin t\,dt, \quad dy = b\cos t\,dt;$$

$$\therefore\ ds = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\,.\,dt,$$

$$\tan\psi = -\frac{b}{a}\cot t,$$

$$\cos\psi = \frac{1}{\sqrt{1+\frac{b^2}{a^2}\cot^2 t}}.$$

Equation to tangent is

$$\frac{x\cos t}{a} + \frac{y\sin t}{b} = 1.$$

(4) To find the locus of the intersection of the tangent and a perpendicular upon it from the focus of an ellipse.

The equation to the tangent is

$$y = -\frac{b}{a}\cot t\,.\,x + \frac{b}{\sin t}.$$

The equation to the perpendicular upon this from the focus is

$$y = \frac{a}{b}\tan t\,.\,(x - ae),$$

which equations may be put in the form

$$ay\sin t + bx\cos t = ab,$$

$$by\cos t - ax\sin t = -a^2e\sin t;$$

therefore, squaring and adding these equations,

$$(x^2+y^2)(a^2\sin^2 t + b^2\cos^2 t) = a^2(b^2 + a^2e^2\sin^2 t)$$

$$= a^2(b^2\cos^2 t + a^2\sin^2 t).$$

Since $a^2e^2 = a^2 - b^2$;

$$\therefore\ x^2 + y^2 = a^2;$$

hence the locus required is the circle described on the major axis as diameter.

(5) To find what relation must hold between α and β in the equation

$$\frac{x_1}{\alpha} + \frac{y_1}{\beta} = 1 \ldots (1).$$

So that it shall be the equation to a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ldots (2).$$

It is easy to shew that the equation to the tangent to (2) is

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

Comparing this with (1) we have

$$\alpha = \frac{a^2}{x}, \quad \beta = \frac{b^2}{y};$$

$$\therefore \frac{x}{a} = \frac{a}{\alpha}, \quad \frac{y}{b} = \frac{b}{\beta};$$

$$\therefore \text{by (2)} \quad \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = 1,$$

which is the relation required.

(6) To find the curve in which the subnormal is a constant c.

We have subnormal $= y\dfrac{dy}{dx} = c$;

$$\therefore 2ydy - 2cdx = 0;$$

$$\therefore d(y^2 - 2cx) = 0.$$

Now in general if $du = 0$, u must be a constant;

$\therefore y^2 = 2cx +$ some constant, which is the equation to a parabola.

(7) To find the curve in which the subtangent is twice the abscissa.

We have $y\dfrac{dx}{dy} = 2x$;

$$\therefore \frac{dx}{x} - 2\frac{dy}{y} = 0;$$

$$\therefore d(\log x - 2\log y) = 0;$$

$$\therefore \log x - 2\log y = \text{constant} = \log C \text{ suppose};$$

$\therefore \dfrac{x}{y^2} = C$, which is the equation to a parabola.

(8) In the ellipse, if n be the normal and p the perpendicular from the center on the tangent,

$$pn = b^2.$$

For $p = \dfrac{ydx - xdy}{ds}$, $n = y\dfrac{ds}{dx}$;

$$\begin{aligned} \therefore pn &= y^2 - xy\frac{dy}{dx} \\ &= y^2 + \frac{b^2x^2}{a^2}, \text{ since } \frac{dy}{dx} = -\frac{b^2x}{a^2y} \\ &= b^2. \end{aligned}$$

(M.)

(1) Let the equation to the curve be

$$a^3y = x^4 - 6ax^3 + 12a^2x^2,$$

then $a^3\dfrac{d^2y}{dx^2} = 12(x^2 - 3ax + 2a^2) = 12(x-a)(x-2a)$;

$\therefore$ the sign of $\dfrac{d^2y}{dx^2}$ is + when x is $< a$,

− when x is $> a$ and $< 2a$,

and + when x is $> 2a$.

Hence the concavity of the curve is turned upwards when x is $< a$, downwards when x is between a and $2a$, and upwards when x is $> 2a$. And there are two points of contrary flexure, one when $x = a$, and another when $x = 2a$.

(2) Let the equation to the curve be

$$y = \frac{a^2 x}{(x-a)^2},$$

$$\text{then } \frac{d^2y}{dx^2} = \frac{2a^2(x+2a)}{(x-a)^4}.$$

Here there is a change of flexure when x passes through the value $-2a$, but none when x passes through the value a.

(N.)

(1) To find ρ in the ellipse.

We have $x = a\cos t$, $y = b\sin t$, {page 217 (3),}

$$\rho = \frac{ds^3}{dx d^2y - dy d^2x}$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}{ab\sin^2 t + ab\cos^2 t}$$

$$= \frac{1}{ab}\left(\frac{a^2}{b^2}y^2 + \frac{b^2}{a^2}x^2\right)^{\frac{3}{2}}.$$

(2) If n be the normal,

$$n = y\frac{ds}{dx};$$

$$\therefore\ ds = \frac{n\,dx}{y} = -\frac{a}{b}n\,dt;$$

$$\therefore\ \rho = -\frac{1}{ab}\cdot\frac{a^3}{b^3}n^3 = -\frac{n^3}{\left(\frac{b^2}{a}\right)^2}.$$

The negative sign indicates that the concavity of the curve is turned downwards, in which case $\frac{d\psi}{ds}$ is negative.

(3) To find ρ in the parabola.

Let us use the equation

$$y = \frac{x^2}{4m},$$

$$\text{then} \quad dy = \frac{x}{2m}\,dx;$$

$$\therefore\ ds^2 = \left(1 + \frac{x^2}{4m^2}\right) dx^2 = \left(1 + \frac{y}{m}\right) dx^2,$$

$$\text{and} \quad d^2y = \frac{dx^2}{2m} \quad (dx \text{ being constant});$$

$$\therefore\ \rho = \frac{ds^3}{dx\,d^2y} = 2m\left(1 + \frac{y}{m}\right)^{\frac{3}{2}}.$$

(O.)

(1) To find the evolute of the ellipse.

We have

$$\alpha = x - \rho \sin\psi,$$

$$\beta = y + \rho\cos\psi,$$

$$\text{or} \quad \alpha = x - \frac{ds^2 dy}{dx\,d^2y - dy\,d^2x},$$

$$\beta = y + \frac{ds^2 dx}{dx\,d^2y - dy\,d^2x}.$$

Hence in the ellipse,

$$\alpha = a\cos t - \frac{(a^2\sin^2 t + b^2\cos^2 t)\,b\cos t}{ab}$$

$$= \frac{1}{a}\,(a^2 - a^2\sin^2 t - b^2\cos^2 t)\cos t$$

$$= \frac{a^2 - b^2}{a}\cos^3 t.$$

Similarly $\beta = \frac{b^2 - a^2}{b} \sin^3 t$.

Hence, since $\sin^2 t + \cos^2 t = 1$, we have

$$\left(\frac{\alpha}{a_1}\right)^{\frac{2}{3}} + \left(\frac{\beta}{b^1}\right)^{\frac{2}{3}} = 1,$$

$$\text{when } a_1 = \frac{a}{a^2 - b^2}, \quad \text{and } b_1 = \frac{b}{b^2 - a^2};$$

which is the equation to the evolute of an ellipse. It is represented by *UVWY*, (fig. 36).

(2) To find the evolute to the parabola.

We have $$y = \frac{x^2}{4m},$$

and supposing dx constant,

$$\alpha = x - \frac{ds^2 dy}{d^2 y\, dx},$$

$$\beta = y + \frac{ds^2}{d^2 y};$$

$$\text{also } ds^2 = \left(1 + \frac{y}{m}\right) dx^2,$$

$$dy = \frac{x}{2m} dx, \quad d^2 y = \frac{dx^2}{2m};$$

$$\therefore \alpha = x - 2m\left(1 + \frac{y}{m}\right)\frac{x}{2m} = -\frac{xy}{m} = -\frac{x^3}{4m^2},$$

$$\beta = y + 2m\left(1 + \frac{y}{m}\right) = 3y + 2m;$$

$$\therefore x = -(4m^2 \alpha)^{\frac{1}{3}}, \quad y = \frac{\beta - 2m}{3};$$

$$\therefore \frac{\beta - 2m}{3} = \frac{(4m^2 \alpha)^{\frac{2}{3}}}{4m},$$

$$\text{or } \beta = 2m + \frac{m^{\frac{1}{3}}}{2^{\frac{2}{3}}} a^{\frac{2}{3}},$$

which is the equation to the evolute of a parabola.

It is represented by *UWV* (fig. 37).

(P.)

(1) Let the equation to the curve be

$$r = ae^{m\theta},$$

$$\text{then} \quad dr = mae^{m\theta}\, d\theta,$$

$$\tan \psi = \frac{r\, d\theta}{dr} = m.$$

Hence in this curve, which is called the logarithmic spiral, the tangent always makes the same angle with the radius vector.

$$ds^2 = r^2\, d\theta^2 + dr^2 = a^2 e^{2m\theta} (1 + m^2)\, d\theta^2;$$

$$\therefore\ ds - a\sqrt{1 + m^2} \,.\, e^{m\theta}\, d\theta = 0,$$

$$d\,(s - a\frac{\sqrt{1 + m^2}}{m} e^{m\theta}) = 0;$$

$$\therefore\ s - a\frac{\sqrt{1 + m^2}}{m} e^{m\theta} = \text{some constant, } C \text{ suppose:}$$

if $s = 0$ when $\theta = 0$, then

$$0 - a\frac{\sqrt{1 + m^2}}{m} = C;$$

$$\therefore\ s = \frac{\sqrt{1 + m^2}}{m} (r - a).$$

(2) The polar equation to an ellipse may be put in the form

$$\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{r^2} \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

$$\therefore \left(\frac{1}{a^2} - \frac{1}{b^2}\right) \cos\theta \sin\theta = \frac{1}{r^2} \cdot \frac{dr}{r\,d\theta} = \frac{1}{r^2} \cot\phi \ldots (2).$$

Now (1) gives us

$$\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \cos^2\theta = \frac{1}{r^2} - \frac{1}{a^2},$$

$$\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \sin^2\theta = \frac{1}{b^2} - \frac{1}{r^2};$$

therefore squaring each side of (2) and substituting these values, we have

$$\left(\frac{1}{r^2} - \frac{1}{a^2}\right)\left(\frac{1}{b^2} - \frac{1}{r^2}\right) = \frac{1}{r^4} \cot^2\phi,$$

$$\text{or} \quad r^4 - (a^2 + b^2)\, r^2 + \frac{a^2 b^2}{\sin^2\phi} = 0,$$

which equation gives us r, the length of the diameter which makes an angle ϕ with the tangent at its extremity.

Hence, since conjugate diameters are parallel to tangents at their extremities, the two values of r got from this equation are the lengths ($a'b'$ suppose) of the two conjugate diameters which make an angle ϕ with each other. We have therefore

$$a'^2 + b'^2 = a^2 + b^2,$$

$$a'^2 . b'^2 = \frac{a^2 b^2}{\sin^2\phi}, \quad \text{or} \quad a'b' \sin\phi = ab,$$

two well known properties of the ellipse.

(3) In the spiral $r = ae^{m\theta}$,

$$p = \frac{r^2 d\theta}{ds} = \frac{r^2 d\theta}{\sqrt{1 + m^2}. r\,d\theta};$$

$$\therefore p = \frac{r}{\sqrt{1 + m^2}} \ldots (1);$$

$$\therefore dp = \frac{dr}{\sqrt{1 + m^2}};$$

$$\therefore \ \rho = \frac{r\,dr}{dp} = \sqrt{1 + m^2}\,.\,r.$$

If O be the extremity of ρ, and $SO = r_1$, then

$$\begin{aligned} r_1^2 &= \rho^2 + r^2 - 2r\rho\cos SPO \\ &= \rho^2 + r^2 - 2p\rho \\ &= m^2r^2\,; \end{aligned}$$

$$\therefore \ r_1 = mr.$$

Also if p_1 be the perpendicular from S on PO, we have

$$p_1^2 + p^2 = r^2\,; \quad \therefore \ p_1 = \frac{m}{\sqrt{1 + m^2}}\,r\,;$$

$$\therefore \ p_1 = \frac{r_1}{\sqrt{1 + m^2}}\,.$$

Now r_1 and p_1 are evidently the radius vector of the evolute and perpendicular upon its tangent; hence, comparing this relation between p_1 and r_1 with (1), it follows that the evolute to a logarithmic spiral is a similar spiral.

In general we may find the equation between p_1 and r_1 by eliminating p, ρ, and r, between the equations

$$\left.\begin{aligned} p &= f(r) \\ \rho &= \frac{r\,dr}{dp} = \frac{r}{f'(r)} \\ r_1^2 &= r^2 + \rho^2 - 2\rho p \\ p_1^2 + p^2 &= r^2 \end{aligned}\right\},$$

$p = f(r)$ being the equation to the given curve between p and r. In this manner we may obtain the equation to the evolute of a spiral.

(S.)

(1) Let the given equation be

$$a^2y^2 = \frac{x^2(x-a)^2(x+2b)}{x+b};$$

$$\therefore\ ay = \pm x(x-a)\sqrt{\frac{x+2b}{x+b}};$$

then we have the following table.

x	y	$\frac{dy}{dx}$
	$\pm$	
$-2b$	0	∞
	imposs.	
$-b$	∞	∞
	$\pm$	
0	0	$\pm\sqrt{2}$
	$\pm$	
a	0	$\pm\sqrt{\frac{a+2b}{a+b}}$
	$\pm$	
∞	∞	∞

Hence the curve is represented by fig. 38.,

$AB = a, \quad AC = b, \quad AD = 2b.$

(2) Let the given equation be

$$y = \frac{a(x-a)^2}{x^2}.$$

See fig. 39., $AB = a$, $AC = a$.

(U.)

(1) To determine the position and nature of the multiple points of the curve,

$$U = y^4 + x^4 - 2a^2y^2 - 2a^2x^2 + a^4 = 0 \ \ldots \ (1),$$

$$\left. \begin{aligned} d_x U &= 4x^3 - 4a^2x = 0 \\ d_y U &= 4y^3 - 4a^2y = 0 \end{aligned} \right\} \ldots\ldots\ldots\ldots \ (2),$$

which equations are satisfied by any two of the values

$$x = 0 \text{ or } \pm a, \quad \text{and } y = 0 \text{ or } \pm a.$$

Now if $x = 0$, $y = \pm a$ in virtue of (1),

and if $x = \pm a$, $y = 0$ or $\pm\sqrt{2} \,.\, a$.

Hence the only values of x and y which satisfy (1) and (2) are

$$\left. \begin{aligned} x &= 0 \\ y &= a \end{aligned} \right\} (\alpha), \quad \left. \begin{aligned} x &= 0 \\ y &= -a \end{aligned} \right\} (\beta), \quad \left. \begin{aligned} x &= a \\ y &= 0 \end{aligned} \right\} (\gamma), \quad \left. \begin{aligned} x &= -a \\ y &= 0 \end{aligned} \right\} (\delta).$$

Now

$$d_x^2 U = 12x^2 - 4a^2,$$

$$d_x d_y U = 0,$$

$$d_y^2 U = 12y^2 - 4a^2.$$

Hence, using the notation in (161), we have

$A = 0, \quad B = 0,$

$C = -4a^2$ if $x = 0$, or $8a^2$ if $x = \pm a$,

$D = 0,$

$E = -4a^2$ if $y = 0$, or $8a^2$ if $x = \pm a$.

Hence for the values (α) and (β), we have

$$-4a^2 + 8a^2u_0^2 = 0, \text{ and } \therefore \ u_0 \text{ or } \frac{dy}{dx} = \pm \frac{1}{\sqrt{2}},$$

and for the values (γ) and (δ), we have

$$8a^2 - 4a^2u_0 = 0, \quad \text{and} \therefore u_0 \text{ or } \frac{dy}{dx} = \pm\sqrt{2}.$$

Hence if we take $AB = AE = AC = AD = a$ (fig. 40), there is a double point at B, at C, at D, and at E, as is represented in the figure.

This is an example which may be very easily solved by the method in (156).

For in this case

$$p = -\frac{x^3 - a^2x}{y^3 - a^2y};$$

$$\therefore p = -\frac{3x^2 - a^2}{(3y^2 - a^2)p};$$

$$\therefore \text{if } x = 0, \quad \text{and } y = \pm a,$$

$$p^2 = \tfrac{1}{2},$$

$$\text{and if } x = \pm a, \quad \text{and } y = 0,$$

$$p^2 = 2.$$

In general the method given in (159) ought to be used, only when we wish to find $\frac{dy}{dx}$ for the values $x = 0$, $y = 0$, or when more than two differentiations are necessary to find $\frac{dy}{dx}$, in which cases it is simpler than the common method.

(2) Let $U = ay^3 - bx^2y + x^4 = 0 \ldots (1)$,

$$\text{then } d_xU = -2bxy + 4x^3 = 0 \;\ldots (2),$$

$$d_yU = 3ay^2 - bx^2 = 0 \;\ldots\ldots\ldots (3).$$

$$(2) \text{ gives } x = 0 \text{ or } y = \frac{2x^2}{b},$$

if $x = 0$ $y = 0$ in virtue of (3) and these values satisfy (1), if $y = \frac{2x^2}{b}$ (3) becomes

$$\frac{12a}{b^2}x^4 - bx^2 = 0,$$

$$\text{which gives } x^2 = \frac{b^3}{12a} \text{ and } \therefore\ y = \frac{b^2}{6a}:$$

these values do not satisfy (1), and are therefore to be rejected. We have therefore to consider only the values $x = 0$, $y = 0$.

Hence putting $y = ux$ in (1), we have

$$au^3 - bu + x = 0,$$

$$\text{and } \therefore\ au_0^3 - bu_0 = 0;$$

$$\therefore\ \frac{dy}{dx} = 0 \text{ or } \pm\sqrt{\frac{b}{a}},$$

which indicates a triple point at the origin.

(3) To examine the nature of the curve

$$y^5 + ax^4 - b^2xy^2 = 0$$

in the immediate vicinity of the origin.

Assume $y = ux^m$, then

$$ux^{5m} + a^5x^4 - b^2u^2x^{2m+1} = 0.$$

Suppose $5m = 4$; $\therefore\ m = \frac{4}{5}$, and $\therefore\ 2m + 1 < 4$; which will not answer.

Suppose $5m = 2m + 1$; $\therefore\ m = \frac{1}{3}$, and $\therefore\ 4 > 2m + 1$; which will answer, and gives

$$u^5 - b^2u^2 = 0, \text{ and } \therefore\ u = b^{\frac{2}{3}}.$$

Suppose $4 = 2m + 1$; $\therefore\ m = \frac{3}{2}$, and $\therefore\ 5m > 4$; which will answer, and gives

$$a - b^2u^2 = 0, \text{ and } \therefore\ u = \pm\frac{\sqrt{a}}{b}.$$

Hence we have

$$y = b^{\frac{2}{3}}x^{\frac{1}{3}} + R,$$

$$\text{and } y = \pm \frac{\sqrt{a}}{b} x^{\frac{3}{2}} + R'.$$

The former of these represents the portion PAP' of the curve (fig. 41), and the latter the portion QAQ'.

We rejected the value $5m = 4$, because it would give $u = \infty$ when $x = 0$; this value belongs to the infinite branches of the curve; for putting $5m = 4$, and $\therefore\ m = \frac{4}{5}$, and assuming $x = \frac{1}{z}$, we have

$$(u^5 + a)\frac{1}{z^4} - b^2 u^2 \frac{1}{z^{\frac{13}{5}}} = 0,$$

$$\text{or } u^5 + a - b^2 u^2 z^{\frac{7}{5}} = 0,$$

which gives the limiting value of $u = -a^{\frac{1}{5}}$ when z approaches 0;

$$\therefore\ y = -a^{\frac{1}{5}} \frac{1}{z^{\frac{4}{5}}} + R$$

$$= -a^{\frac{1}{5}} x^{\frac{4}{5}} + R,$$

$$\text{where } R = 0, \text{ when } x = \infty;$$

$$\therefore\ y = -a^{\frac{1}{5}} x^{\frac{4}{5}} \text{ very nearly for large values of } x;$$

$$\therefore\ \frac{dy}{dx} = 0 \text{ when } x = \infty, \text{ as is represented in the figure.}$$

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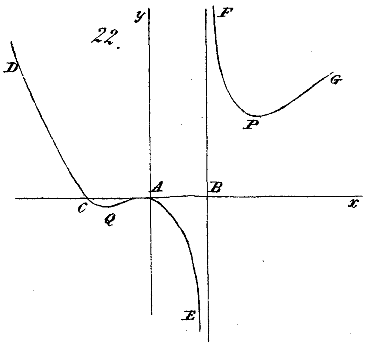

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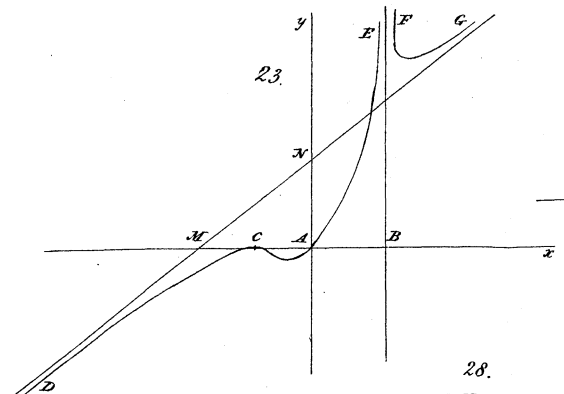

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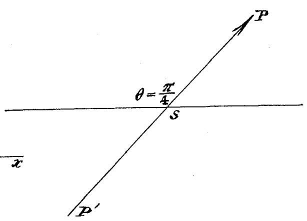

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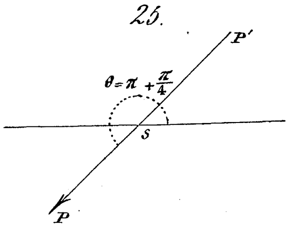

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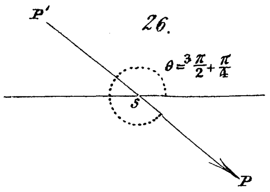

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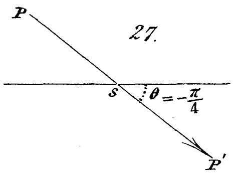

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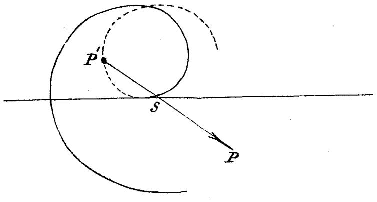

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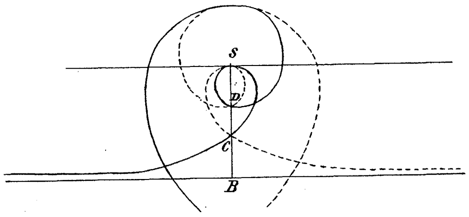

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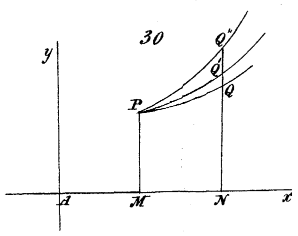

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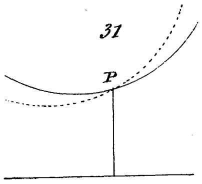

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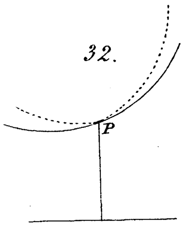

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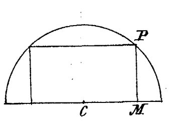

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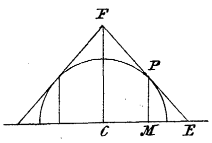

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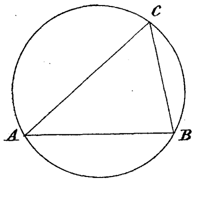

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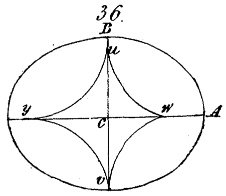

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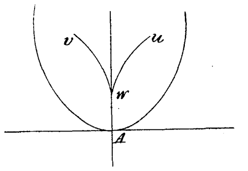

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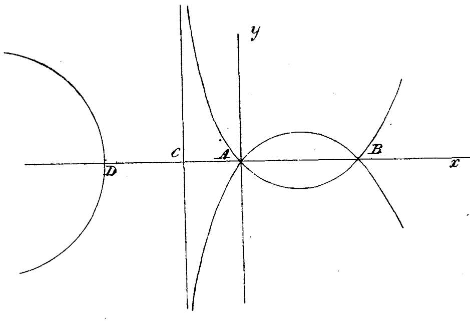

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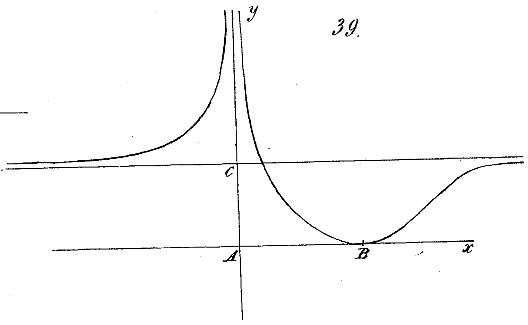

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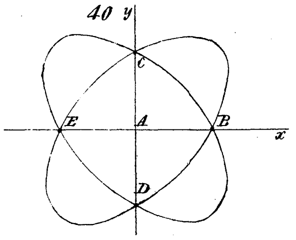

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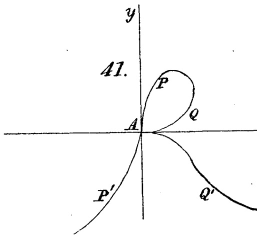

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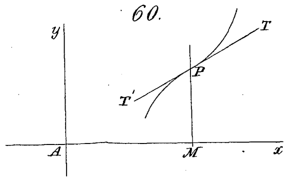

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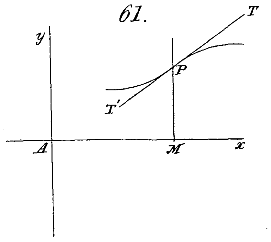

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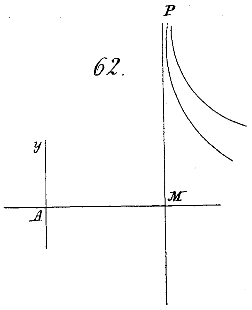

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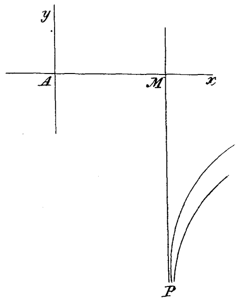

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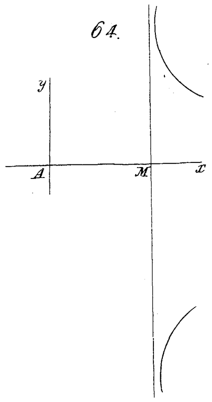

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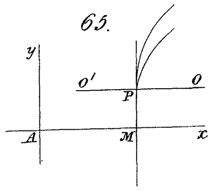

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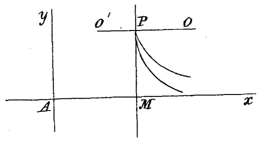

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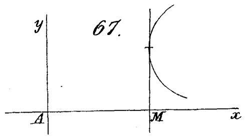

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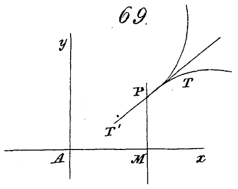

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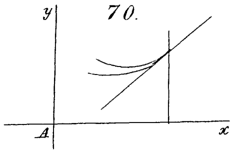

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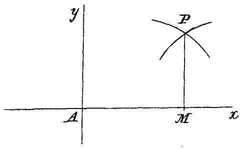

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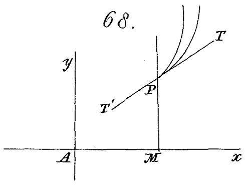

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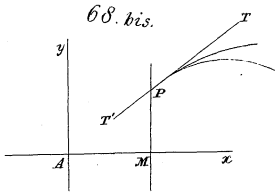

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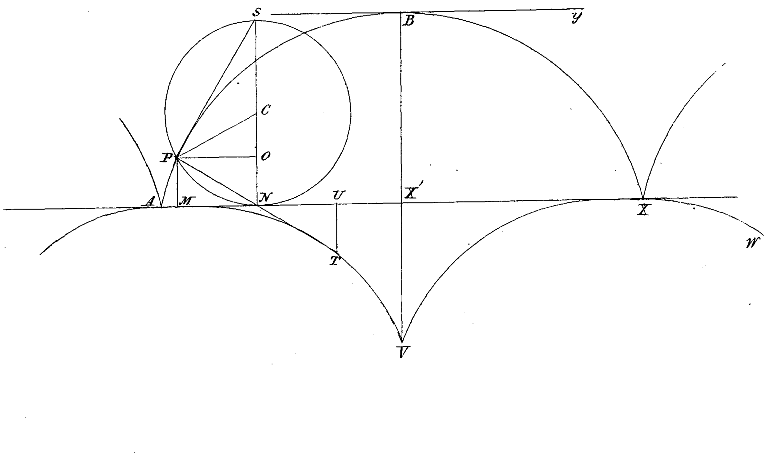

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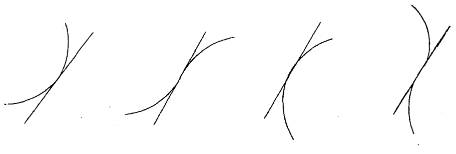

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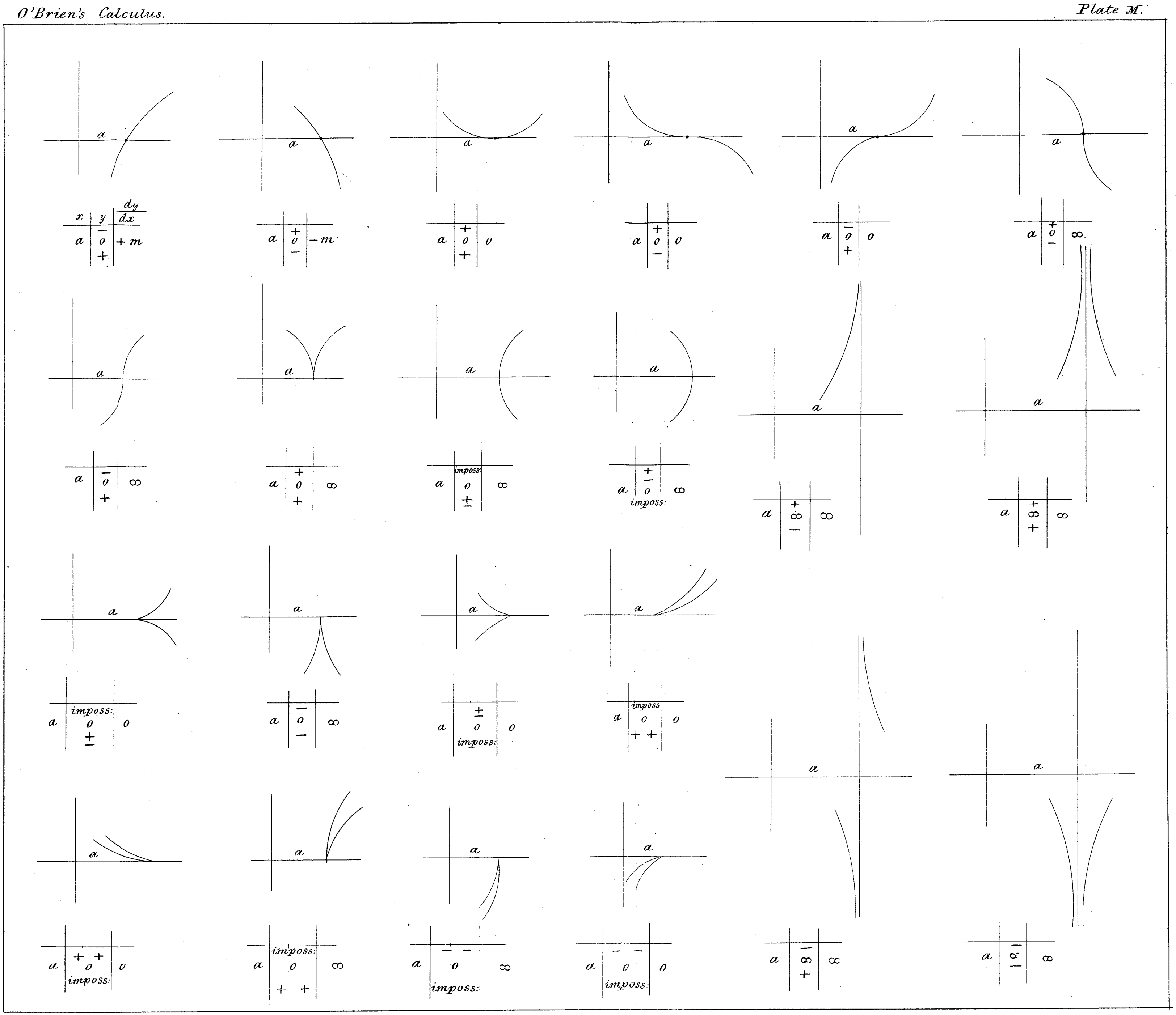

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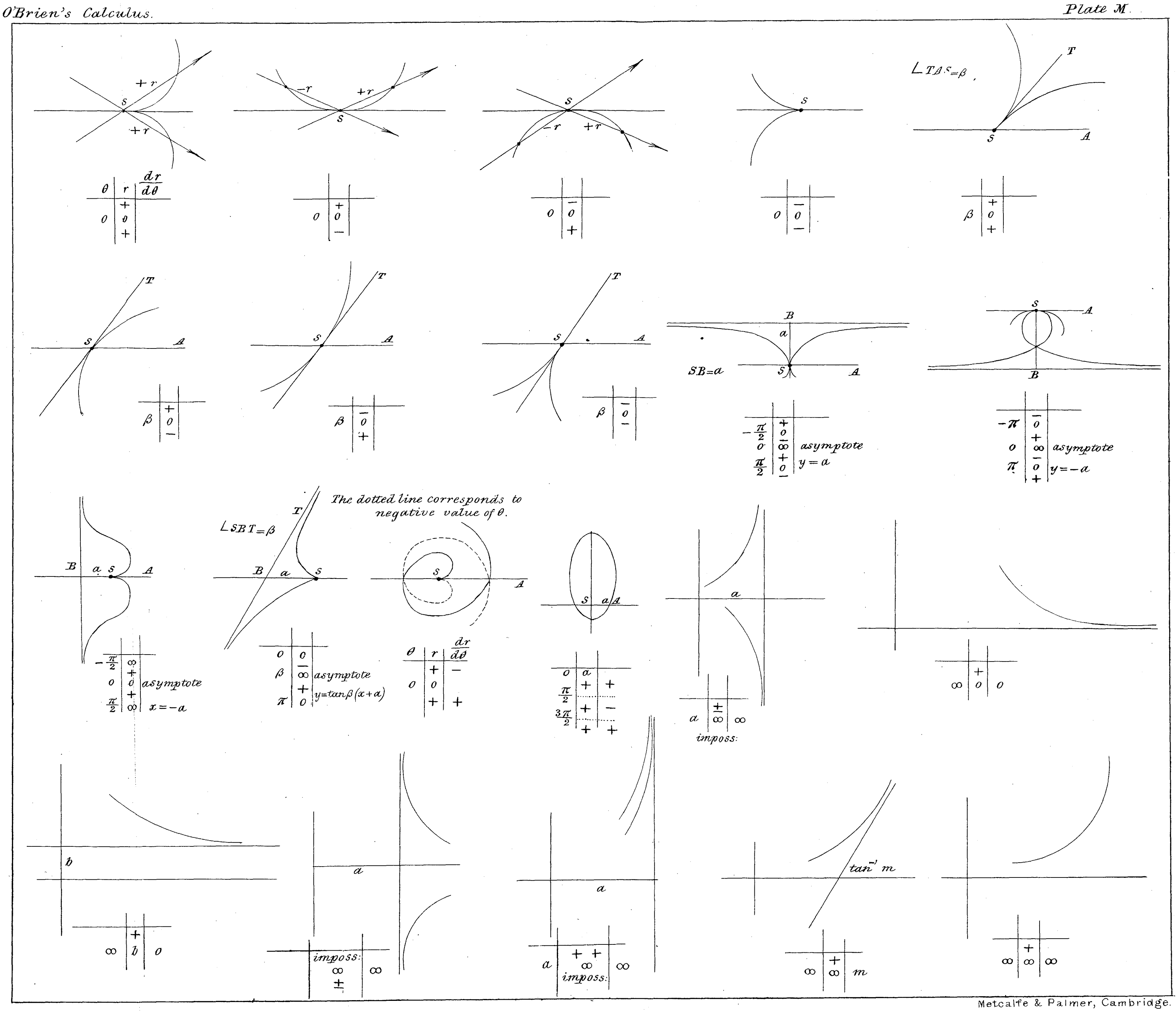

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A DIGESTED

SERIES OF EXAMPLES

IN THE

APPLICATIONS OF THE PRINCIPLES

OF THE

DIFFERENTIAL CALCULUS.

DESIGNED FOR

THE USE OF STUDENTS

In the University.

BY JOHN HIND, M.A. F.C.P.S. F.R.A.S.

LATE FELLOW AND TUTOR OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

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M.DCCC.XXXII.

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THE Collection of Theorems and Problems, submitted to the notice of the public in the following pages, is intended for Practice in the Applications of the "*Principles of the Differential Calculus*" as laid down in the Author's Work upon that subject; and they have been distributed into Chapters agreeably to the plan there pursued, with occasional references to the particular Articles upon which their solutions depend. Although many original examples are introduced, it may be necessary here to observe that by far the greater portion of them have been selected from the Works of different Authors who have written upon the subject, and from Papers proposed at the public and private examinations in the University. The order observed in the arrangement of each Chapter is generally the same as that of the Articles in the corresponding Chapter of the Work referred to, each individual question comprised in every Chapter being supposed capable of solution independently of those which succeed it: and to the whole has been prefixed a Summary of the principal Articles contained in that Work, for the purpose of enabling the Student to recur with greater facility to the Exposition of any Principle that may be required.

CAMBRIDGE,
Feb. 20, 1832.

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CORRECTIONS

To be made in the PRINCIPLES *of the* DIFFERENTIAL CALCULUS.

Page

9, line 4 from bottom, *for* log (4), *read* by (4).

12, ... 2 from bottom, *for* $4\pi r . AN = 4\pi AP^2$, *read* $2\pi r AN = \pi AP^2$.

17, ... 15 from top, *for* Intranscendental, *read* Interscendental, and the same in line 6 from the top of the next page.

87, ... 4 from top, *for* $\frac{d^3u}{dx^3}\frac{h^2}{1.2.3} + \&c. + \frac{d^nu}{dx^n}\frac{h^2}{1.2.3.\&c.n}$,

read $\frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c. + \frac{d^nu}{dx^n}\frac{h^n}{1.2.3.\&c.n}$.

89, ... 2 from top, *for* a^x, *read* a^{x+h}.

134, ... 7 from top, *for* $u\frac{\frac{d^2P}{dx^2}}{\frac{d^2Q}{dx^2}}$, *read* $u = \frac{\frac{d^2P}{dx^2}}{\frac{d^2Q}{dx^2}}$.

195, ... 10 and 13 from top, Cubical and Semi-cubical *must change places.*

195, ... 4 from bottom, *for* $AT = -\frac{m+1}{m}x$, *read* $AT = -(m+1)x$, and in the next two lines, *for* $AW = (m+1)y$, *read* $AW = \frac{m+1}{m}y$, and *for* $LW = -my$, *read* $LW = -\frac{y}{m}$.

200, ... 2 from bottom, *for* $x = 0$, *read* $x' = 0$, and *for* axis of x, *read* axis of y.

224, ... 1 from top, *for* Orders of Contact, *read* Contact and Osculation.

233, ... 9 from bottom, *for* even *read* odd, and in the next line but one, *for* odd *read* even.

267, ... 1 from top, *for* a^9, *read* a^2.

270, ... 2 from top, *for* $Px - SP \cos SPG$, *read* $Px = SP \cos SPG$.

383, ... 5 from top, *for* $\frac{x}{\sqrt{x^2-y^2}}$, *read* $\frac{1}{\sqrt{x^2-y^2}}$, and in line 3 from bottom *for* $-\sec^2\frac{x}{y}$, *read* $-x\sec^2\frac{x}{y}$.

409, ... 4 from top, *for* 2^4, *read* 2^6.

417, ... 8 from bottom, *for* $be - ag$, *read* $ag - be$.

429, ... 6 from bottom, *for* xz and yz, *read* yz and xz.

442, ... 7 from bottom, *for* $XAM = \theta$ and $ZAP = \phi$, *read* $XAM = \phi$ and $ZAP = \theta$; and in the next line but two, *for* $r d\phi$, *read* $r d\theta$, and *for* $\varrho \sin\theta d\theta$, *read* $\varrho \sin\theta d\phi$.

447, ... 2 from top, *for* $1 + P^2 + (z' - \gamma) R$, *read* $1 + P^2 + (z' - \gamma) R = 0$.

489, ... 10 from top, *for* m^{m-4}, *read* x^{m-4}.

493, ... 5 from top, *for* $\cos 5\theta$, *read* $\sin 5\theta$.

496, ... 6 from top, *for* $(a\tan\theta + b^2)$, *read* $(a\tan\theta + b)^2$.

499, ... 3 from top, *for* $\frac{du}{dx}$, *read* $\frac{du}{d\theta}$.

519, ... 10 from bottom, *for* Cube, *read* Parallelopiped, and the same throughout the Article.

524, ... 7 from bottom, *for* tangent, *read* normal, and the same in the next line.

EXAMPLES

IN THE

DIFFERENTIAL CALCULUS.

INTRODUCTORY CHAPTER.

1. WHAT are the inferior and superior limits of

$$\frac{a+bx}{cx+d},\ \frac{ax^2+bx-c}{x^2-ex+f}\ \text{and}\ \frac{a^x+b}{c^x-e}?$$

Answer. $\frac{a}{d}$ and $\frac{b}{c}$; $-\frac{c}{f}$ and a; $\frac{1+b}{1-e}$ and ∞ or 0 accord-as a is greater or less than c.

2. The convex surface of a right cylinder is equal to the area of a circle whose radius is a mean proportional between the height and the diameter of the base.

3. In a right cylinder, the radius of the base : twice the height :: the base of the cylinder : its convex surface.

4. The convex surfaces of right cylinders are to one another in the compound ratios of their heights, and the diameters of their bases.

5. The contents of right cylinders are to one another in the compound ratios of their heights and the areas of their bases.

6. The convex surface of a right cone is equal to the area of a circle whose radius is a mean proportional between the length of the side and the radius of the base.

7. The convex surfaces of right cones are to one another in the compound ratios of the lengths of their sides and the diameters of their bases.

8. The convex surface of a right cylinder : the convex surface of a right cone of the same base and altitude :: height of the cylinder : half the side of the cone.

9. The convex surface of the frustum of a right cone is equal to a circle whose radius is a mean proportional between the side and the sum of the radii of the bases.

10. The contents of cones are to one another in the compound ratios of their heights and the areas of their bases.

11. The surface of a sphere is equal to the convex surface of its circumscribed cylinder.

12. If a sphere and its circumscribed cylinder be cut by two planes parallel to the base, the intercepted parts of the surfaces of the sphere and cylinder will be equal.

13. The whole surface of a hemisphere is equal to three times the area of its base.

14. The convex surfaces of spherical segments are as the corresponding segments of the diameters and the diameters jointly.

15. The surface and solidity of a sphere are each two thirds of those of the circumscribed cylinder : required a proof.

16. The surface of a sphere is twice as great as the convex surface of the inscribed equilateral cylinder.

17. The whole surface of the inscribed equilateral cylinder is three-fourths of the surface of the sphere.

18. The whole surface of a cylinder circumscribing a sphere is twice as great as that of the inscribed equilateral cylinder.

19. The convex surface of any segment of a sphere is to the convex surface of the inscribed right cone on the same base as the side of the cone to the radius of the base. Compare them also for the hemisphere.

20. If the inscribed cone be equilateral, the surface of the spherical segment is twice as great as the convex surface of the cone.

21. If a right-angled cone circumscribe a hemisphere, the surface of the hemisphere : the convex surface of the cone :: 1 : $\sqrt{2}$.

22. The surface of a right-angled cone circumscribed about a hemisphere is twice as great as that of the inscribed cone, whether the convex or whole surfaces be taken.

23. The surface of a sphere : the convex surface of an equilateral inscribed cone :: 8 : 3.

24. The whole surface of an equilateral cone inscribed in a sphere : the surface of the sphere :: 3^2 : 4^2.

25. The surface of a sphere : the whole surface of the circumscribed equilateral cone :: 2^2 : 3^2.

26. A sphere is equal to a cone whose height is the radius and base equal to four great circles of the sphere.

27. A hemisphere is twice as great as its inscribed cone.

28. A right cone, sphere and cylinder of the same base and altitude are as the numbers 1, 2, 3.

29. Sectors of spheres are to one another in the compound ratios of their radii and convex surfaces.

30. Find the content and surface of the middle zone of a sphere.

31. If the altitude of a cylinder be equal to the diameter of its base, the whole surface is equal to six times the area of the base, required a proof.

32. If the bases of a cylinder and of a cone have the same radius as a sphere, and each of their altitudes equal to the diameter of the sphere, the solidity of the cone is equal to the excess of the cylinder above the sphere.

33. The content of a hollow sphere is equal to the frustrum of a cone whose bases are equal to the exterior and interior surfaces of the solid, and height is equal to the difference of their radii.

34. A sphere has to its inscribed equilateral cylinder, its inscribed equilateral cone and its circumscribed equilateral cone, the ratios $4\sqrt{2} : 3$, $32 : 9$ and $4 : 9$ respectively.

35. The right cylinder and equilateral cone circumscribing a sphere are to one another as $2 : 3$, whether we consider their convex surfaces, whole surfaces, contents, bases or heights.

36. It is required to divide the area of a given circle into n equal parts by means of concentric circles.

37. Divide the area of a circle into n equal parts whose perimeters shall be equal.

38. Divide a given sphere into n equal parts by means of concentric spherical surfaces.

39. The whole surface of a cone is three times as great as the area of the base; find the vertical angle.

40. Find the radius of the sphere which can be inscribed in a given cone.

41. Compare the sum of the contents of all the spheres that can be inscribed in a cone, having their centres in its axis, with that of the cone itself.

42. Divide a right cone into two parts, whose contents shall be in the ratio of $m : 1$, by a plane parallel to its base. Also, that their convex surfaces may be in the same ratio.

CHAP. I.

MISCELLANEOUS EXAMPLES.

PROVE from first Principles the truth of the following formulæ, as in the examples of Article (18).

1. If $u = ax^2 + bx + c$, then $\dfrac{du}{dx} = 2ax + b$.

2. If $u = \dfrac{a}{x^2} - \dfrac{b}{x} + cx$, then $\dfrac{du}{dx} = -\dfrac{2a}{x^3} + \dfrac{b}{x^2} + c$.

3. If $u = ax^{\frac{3}{2}} - bx^{\frac{1}{2}} + c$, then $\dfrac{du}{dx} = \dfrac{3}{2}ax^{\frac{1}{2}} - \dfrac{1}{2}bx^{-\frac{1}{2}}$.

4. If $u = a^2(b + x)^2 - b^2(a - x)^2$, then

$$\frac{du}{dx} = 2(a + b)\left\{ab + (a - b)x\right\}.$$

5. If $u = \sqrt{1 - x^2}$, then $\dfrac{du}{dx} = \dfrac{-x}{\sqrt{1 - x^2}}$.

6. If $u = \sqrt{ax^2 + 2bx + 3c}$, $\dfrac{du}{dx} = \dfrac{ax + b}{\sqrt{ax^2 + 2bx + 3c}}$.

7. If $u = \dfrac{x}{\sqrt{1 + x^2}}$, then $\dfrac{du}{dx} = \dfrac{1}{(1 + x^2)^{\frac{3}{2}}}$.

8. If $u = \dfrac{x^2}{\sqrt{a^3 + x^3}}$, then $\dfrac{du}{dx} = \dfrac{x(4a^3 + x^3)}{2(a^3 + x^3)^{\frac{3}{2}}}$.

9. If $u = \dfrac{e^x}{e^x + 1}$, then $\dfrac{du}{dx} = \dfrac{e^x}{(e^x + 1)^2}$.

10. If $u = \sin x . \cos 2x$, $\dfrac{du}{dx} = \cos x \cos 2x - 2 \sin x \sin 2x$.

CHAP. II.

MISCELLANEOUS EXAMPLES.

By means of the rules laid down and observations made in Articles (20)—(33), it is required to prove the following formulæ.

1. If $u = ax^3 - bx^2 + c$, then $\frac{du}{dx} = 3ax^2 - 2bx$.

2. If $u = ax^m - bx^{m-1} + cx^{m-2} + c$, then

$$\frac{du}{dx} = max^{m-1} - (m-1)\,bx^{m-2} + (m-2)\,cx^{m-3}.$$

3. If $u = (2n-1)\,x^{2n+1} - (2n+1)\,x^{2n-1}$, then

$$\frac{du}{dx} = (4n^2 - 1)\,x^{2n-2}\,(x^2 - 1).$$

4. If $u = (ax^{2n} - bx^n + c)^m$, then

$$\frac{du}{dx} = mn\,(ax^{2n} - bx^n + c)^{m-1}\,(2ax^n - b)\,x^{n-1}.$$

5. If $u = (1-x)\,(1+x^2)$, $\frac{du}{dx} = -(1 - 2x + 3x^2)$.

6. If $u = (x-1)^4\,(x+2)^3$, then

$$\frac{du}{dx} = (7x+5)\,(x-1)^3\,(x+2)^2.$$

7. If $u = (a+x)\,(b+2x)\,(c+3x)$, then

$$\frac{du}{dx} = 18x^2 + (12a + 6b + 4c)\,x + 3ab + 2ac + bc.$$

8. If $u = \frac{a+x}{b+x}$, then $\frac{du}{dx} = -\frac{a-b}{(b+x)^2}$.

9. If $u = \frac{x^3}{a^2} + \frac{a^2}{x}$, then $\frac{du}{dx} = 3\left(\frac{x}{a}\right)^2 - \left(\frac{a}{x}\right)^2$.

10. If $u = \frac{x^m - a^m}{x^m + a^m}$, then $\frac{du}{dx} = \frac{2 m a^m x^{m-1}}{(x^m + a^m)^2}$.

11. If $u = \frac{3x^2 - a^2}{(a^2 + x^2)^3}$, then $\frac{du}{dx} = \frac{12x(a^2 - x^2)}{(a^2 + x^2)^4}$.

12. If $u = \frac{a + 5bx}{(a + bx)^5}$, then $\frac{du}{dx} = -\frac{20b^2 x}{(a + bx)^6}$.

13. If $u = \frac{a + 3bx^2}{(a + bx^2)^3}$, then $\frac{du}{dx} = -\frac{12b^2 x^3}{(a + bx^2)^4}$.

14. If $u = ax^{\frac{3}{2}} - bx^{\frac{1}{2}} + cx^{-\frac{1}{2}} - ex^{-\frac{5}{2}}$, then

$$\frac{du}{dx} = \tfrac{1}{2}\left\{3ax^{\frac{1}{2}} - bx^{-\frac{1}{2}} - cx^{-\frac{3}{2}} + 5ex^{-\frac{7}{2}}\right\}.$$

15. If $u = \left\{a + bx^{\frac{3}{2}} + cx^{\frac{2}{3}}\right\}^{\frac{1}{2}}$, $\frac{du}{dx} = \frac{9bx^{\frac{1}{2}} + 4cx^{-\frac{1}{3}}}{12\left\{a + bx^{\frac{3}{2}} + cx^{\frac{2}{3}}\right\}^{\frac{1}{2}}}$.

16. If $u = \left\{1 - x^{\frac{1}{2}} + x^{\frac{1}{3}}\right\}^{\frac{4}{5}}$, $\frac{du}{dx} = \frac{2}{15}x^{-\frac{7}{6}}\left\{\frac{2x^{\frac{1}{2}} - 3x^{\frac{2}{3}}}{(1 - x^{\frac{1}{2}} + x^{\frac{1}{3}})^{\frac{1}{5}}}\right\}$.

17. If $u = \frac{4}{5}(3x - 2)(1 + x)^{\frac{3}{2}}$, $\frac{du}{dx} = 6x\sqrt{1 + x}$

18. If $u = (a + x)\sqrt{a - x}$, $\frac{du}{dx} = \frac{a - 3x}{2\sqrt{a - x}}$.

19. If $u = (a^2 + x^2)\sqrt{a^2 - x^2}$, $\frac{du}{dx} = \frac{x(a^2 - 3x^2)}{\sqrt{a^2 - x^2}}$.

20. If $u = (2a^2 + 3x^2)(a^2 - x^2)^{\frac{3}{2}}$, $\frac{du}{dx} = -15x^3\sqrt{a^2 - x^2}$.

21. If $u = (2a^{\frac{1}{2}} + x^{\frac{1}{2}})\sqrt{a^{\frac{1}{2}} + x^{\frac{1}{2}}}$, $\frac{du}{dx} = \frac{1}{2\sqrt{x}}\,\frac{3a^{\frac{1}{2}} + 2x^{\frac{1}{2}}}{\sqrt{a^{\frac{1}{2}} + x^{\frac{1}{2}}}}$.

22. If $u = (bx - 3a)(a + bx)^{\frac{1}{3}}$, $\frac{du}{dx} = \frac{4b^2 x}{3(a + bx)^{\frac{2}{3}}}$.

23. If $u=(3bx-2a)(a+bx)^{\frac{3}{2}}$, $\dfrac{du}{dx}=\dfrac{15}{2}b^2x\sqrt{a+bx}$

24. If $u=(5bx^2-2a)(a+bx^2)^{\frac{5}{2}}$, $\dfrac{du}{dx}=35b^2x^3(a+bx^2)^{\frac{3}{2}}$.

25. If $u=(8a^2-4abx+3b^2x^2)\sqrt{a+bx}$, $\dfrac{du}{dx}=\dfrac{15b^3x^2}{2\sqrt{a+bx}}$

26. If $u=x(a^2+x^2)\sqrt{a^2-x^2}$, $\dfrac{du}{dx}=\dfrac{a^4+a^2x^2-4x^4}{\sqrt{a^2-x^2}}$.

27. If $u=\dfrac{x}{\sqrt{a-bx^2}}$, then $\dfrac{du}{dx}=\dfrac{a}{(a-bx^2)^{\frac{3}{2}}}$.

28. If $u=\dfrac{\sqrt{a^2-x^2}}{x}$, then $\dfrac{du}{dx}=-\dfrac{a^2}{x^2\sqrt{a^2-x^2}}$.

29. If $u=\dfrac{x^{\frac{3}{2}}}{\sqrt{a-x}}$, then $\dfrac{du}{dx}=\dfrac{(3a-2x)\,x^{\frac{1}{2}}}{2(a-x)^{\frac{3}{2}}}$.

30. If $u=\dfrac{(a+bx^2)^{\frac{3}{2}}}{x^3}$, then $\dfrac{du}{dx}=-\dfrac{3a(a+bx^2)^{\frac{1}{2}}}{x^4}$.

31. If $u=\dfrac{2a+bx}{\sqrt{a+bx}}$, then $\dfrac{du}{dx}=\dfrac{b^2x}{2(a+bx)^{\frac{3}{2}}}$.

32. If $u=\dfrac{a+2bx}{\sqrt{ax+bx^2}}$, then $\dfrac{du}{dx}=-\dfrac{a^2}{2(ax+bx^2)^{\frac{3}{2}}}$.

33. If $u=\dfrac{(x+a)^{\frac{3}{2}}}{\sqrt{x-a}}$, then $\dfrac{du}{dx}=\dfrac{x-2a}{x-a}\sqrt{\dfrac{x+a}{x-a}}$.

34. If $u=\dfrac{x^4+4a^2x^2-8a^4}{\sqrt{a^2-x^2}}$, then $\dfrac{du}{dx}=-\dfrac{3x^5}{(a^2-x^2)^{\frac{3}{2}}}$.

35. If $u=\dfrac{3ax+2bx^3}{(a+bx^2)^{\frac{3}{2}}}$, then $\dfrac{du}{dx}=\dfrac{3a^2}{(a+bx^2)^{\frac{5}{2}}}$.

36. If $u=\frac{b+2cx}{\sqrt{a+bx+cx^2}}$, $\frac{du}{dx}=\frac{4ac-b^2}{2\,(a+bx+cx^2)^{\frac{3}{2}}}$.

37. If $u=\frac{2x^2-a^2}{x^3}\sqrt{a^2+x^2}$, $\frac{du}{dx}=\frac{3a^4}{x^4\sqrt{a^2+x^2}}$.

38. If $u=\frac{x-1}{x+1}\sqrt{x^4+1}$, $\frac{du}{dx}=\frac{2\,(x^5+x^4-x^3+1)}{(x+1)^2\sqrt{x^4+1}}$.

39. If $u=\frac{2x^2-a^2}{x\sqrt{a^2+x^2}}$, $\frac{du}{dx}=\frac{a^2\,(a^2+4x^2)}{x^2\,(a^2+x^2)^{\frac{3}{2}}}$.

40. If $u=\frac{a^3-x^3}{(a^2-x^2)^{\frac{3}{2}}}$, $\frac{du}{dx}=\frac{3a^2x}{(a+x)\sqrt{(a^2-x^2)^3}}$.

41. If $u=\frac{\sqrt{2ax-x^2}}{\sqrt{a^2-x^2}}$, $\frac{du}{dx}=\frac{a}{(a^2-x^2)^{\frac{3}{2}}}\,\frac{a^2-ax+x^2}{\sqrt{2ax-x^2}}$.

42. If $u=\frac{\sqrt{a^2+x^2}}{\sqrt[3]{a^3+x^3}}$, $\frac{du}{dx}=\frac{a^2x\,(a-x)}{\sqrt{a^2+x^2}\,(a^3+x^3)^{\frac{4}{3}}}$.

43. If $u=x\sqrt{\frac{1+x^2}{1-x^2}}$, $\frac{du}{dx}=\frac{1+2x^2-x^4}{(1-x^2)\sqrt{1-x^4}}$.

44. If $u=\left\{5\,(x-1)^3+21\,(x-1)^2+35x\right\}\sqrt{x-1}$, then

$$\frac{du}{dx}=\frac{35x^3}{2\sqrt{x-1}}.$$

45. If $u=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$, then

$$\frac{du}{dx}=-\frac{2x}{\sqrt{1-x^4}\,(1-\sqrt{1-x^4})}.$$

46. If $ux-ax^2-bu^2=0$, then $\frac{du}{dx}=\frac{u-2ax}{2bu-x}$.

47. If $ux(a-x)=a^3$, then $\dfrac{du}{dx}=-\dfrac{u(a-2x)}{x(a-x)}$.

48. If $u^3-ux^2-x^3=0$, then $\dfrac{du}{dx}=\dfrac{x(2u+3x)}{3u^2-x^2}$.

49. If $(u^2+x^2)^3-4a^2u^2x^2=0$, then

$$\frac{du}{dx}=-\frac{x}{u}\left\{\frac{3u^4+(6x^2-4a^2)u^2+3x^4}{3u^4+(6x^2-4a^2)x^2+3x^4}\right\}.$$

50. If $u^2+x^2=2a\sqrt{x^2-u^2}$, then

$$\frac{du}{dx}=\frac{x}{u}\left\{\frac{a-\sqrt{x^2-u^2}}{a+\sqrt{x^2-u^2}}\right\}.$$

51. If $ux=(a+u)\sqrt{b^2-u^2}$, then

$$\frac{du}{dx}=-\frac{u(a+u)(b^2-u^2)}{x(ab^2+u^3)}.$$

52. If $u=x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+$ &c. to n terms, then

$$\frac{du}{dx}=1+x+x^2+x^3+\text{\&c. to } n \text{ terms}=\frac{x^n-1}{x-1}.$$

53. If $u=\sqrt{x^2+a^2+\sqrt{x^2+a^2+\sqrt{\&c.\ \textit{in inf.}}}}$: then

$$\frac{du}{dx}=\pm\frac{2x}{\sqrt{4x^2+4a^2+1}}.$$

54. If $u=\sqrt{x^2+2ax+\sqrt{x^2+2ax+\sqrt{\&c.\ \textit{in inf.}}}}$:

$$\frac{du}{dx}=\pm\frac{2(x+a)}{\sqrt{4x^2+8ax+1}}.$$

CHAP. III.

MISCELLANEOUS EXAMPLES.

It is required by means of the Articles (34)–(46), to deduce the following results.

1. If $u = a^x - \frac{1}{a^x}$, then $\frac{du}{dx} = \left(a^x + \frac{1}{a^x}\right) \log a$.

2. If $u = (x-1)\, a^x$, $\frac{du}{dx} = a^x x \log a - (\log a - 1)\, a^x$.

3. If $u = e^x x^m$, then $\frac{du}{dx} = e^x x^{m-1}(m + x)$.

4. If $u = e^x(x^3 - 3x^2 + 6x - 6)$, $\frac{du}{dx} = x^3 e^x$.

5. If $u = \frac{a^x}{a^{2x} - 1}$, $\frac{du}{dx} = -\log a \; a^x \, \frac{a^{2x} + 1}{(a^{2x} - 1)^2}$.

6. If $u = \frac{x-1}{x+1} e^x$, $\frac{du}{dx} = \frac{(x^2+1)\, e^x}{(x+1)^2}$.

7. If $u = e^x \sqrt{x^2+1}$, $\frac{du}{dx} = \frac{x^2 + x + 1}{\sqrt{x^2+1}} e^x$.

8. If $u = \sqrt{\frac{e^x - 1}{e^x + 1}}$, $\frac{du}{dx} = \frac{e^x}{(e^x+1)\sqrt{e^{2x} - 1}}$.

9. If $u = e^x \sqrt{\frac{1+x}{1-x}}$, $\frac{du}{dx} = \frac{(2 - x^2)\, e^x}{(1-x)\sqrt{1-x^2}}$.

10. If $u = e^x \sqrt{\dfrac{x^2+1}{x^2-1}}$, then $\dfrac{du}{dx} = \dfrac{(x^4-2x-1)\,e^x}{(x^2-1)\sqrt{x^4-1}}$.

11. If $u = e^x \sqrt{\dfrac{x^2+x+1}{x^2-x+1}}$, then

$$\frac{du}{dx} = \frac{(x^4+2)\,e^x}{(x^2-x+1)\sqrt{x^4+x^2+1}}.$$

12. If $u = 2e^{\sqrt{x}}\,(x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6)$, then $\dfrac{du}{dx} = xe^{\sqrt{x}}$.

13. If $u = \dfrac{\sqrt{e^x+1} + \sqrt{e^x-1}}{\sqrt{e^x+1} - \sqrt{e^x-1}}$, $\dfrac{du}{dx} = e^x + \dfrac{e^{2x}}{\sqrt{e^{2x}-1}}$.

14. If $u = 2\log(x+2) - \log(x+1)$, $\dfrac{du}{dx} = \dfrac{x}{x^2+3x+2}$.

15. If $u = \log\left(\dfrac{1}{x} + \dfrac{1}{x^2}\right)$, then $\dfrac{du}{dx} = -\dfrac{x+2}{x(x+1)}$.

16. If $u = \log(x+x^5) - \dfrac{3}{2}\log(1+x^4)$, $\dfrac{du}{dx} = \dfrac{1-x^4}{x(1+x^4)}$.

17. If $u = \log\dfrac{(x+4)^2}{x+2}$, then $\dfrac{du}{dx} = \dfrac{x}{x^2+6x+8}$.

18. If $u = \log\left(\dfrac{x-2}{x+1}\right) + 3\log\left(\dfrac{x}{x-1}\right)$, then

$$\frac{du}{dx} = \frac{6}{x^4-2x^3-x^2+2x}.$$

19. If $u = (x^2-a^2)\log\left(\dfrac{a+x}{a-x}\right) + 2ax$, then

$$\frac{du}{dx} = 2x\log\left(\frac{a+x}{a-x}\right).$$

20. If $u = \frac{a}{a+x} + \frac{b}{b+x} - \frac{a+b}{a-b} \log \left(\frac{a+x}{b+x}\right)$, then

$$\frac{du}{dx} = \frac{(a-b)^2 x}{(a+x)^2 (b+x)^2}.$$

21. If $u = x^3 \left\{(\log x)^3 - (\log x)^2 + \frac{2}{3} \log x - \frac{2}{9}\right\}$, then

$$\frac{du}{dx} = 3x^2 (\log x)^3.$$

22. If $u = \log (x + \sqrt{x^2+1})$, then $\frac{du}{dx} = \frac{1}{\sqrt{x^2+1}}$.

23. If $u = \log (x + a + \sqrt{2ax + x^2})$, $\frac{du}{dx} = \frac{1}{\sqrt{2ax + x^2}}$.

24. If $u = \log (2x - 1 + 2\sqrt{x^2 - x - 1})$, $\frac{du}{dx} = \frac{1}{\sqrt{x^2 - x - 1}}$.

25. If $u = \log \sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$, then $\frac{du}{dx} = -\frac{2a^2 x}{a^4 - x^4}$.

26. If $u = \log \sqrt[3]{\frac{x^3+1}{x^3-1}}$, then $\frac{du}{dx} = -\frac{2x^2}{x^6 - 1}$.

27. If $u = \log \left(\frac{x - \sqrt{1-x^2}}{\sqrt{1-x^2}}\right)$, $\frac{du}{dx} = \frac{x + \sqrt{1-x^2}}{(2x^2-1)(1-x^2)}$.

28. If $u = \log \left(\frac{1 + \sqrt[3]{x}}{1 - \sqrt[3]{x}}\right)$, then $\frac{du}{dx} = \frac{2}{3x^{\frac{2}{3}}(1 - x^{\frac{2}{3}})}$.

29. If $u = \log \frac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a}$, then $\frac{du}{dx} = \frac{2a}{x\sqrt{a^2 + x^2}}$.

30. If $u = \log \frac{\sqrt{x^2 + a^2} - x}{\sqrt{x^2 + a^2} + x}$, then $\frac{du}{dx} = -\frac{2}{\sqrt{x^2 + a^2}}$.

31. If $u = \log \dfrac{x^{\frac{3}{2}} + \sqrt{x^3 - 1}}{x^{\frac{3}{2}} - \sqrt{x^3 - 1}}$, then $\dfrac{du}{dx} = \dfrac{3x^{\frac{1}{2}}}{\sqrt{x^3 - 1}}$.

32. If $u = \log \dfrac{\sqrt{x^2 + 4x + 3}}{x + 2}$, $\dfrac{du}{dx} = \dfrac{1}{x^3 + 6x^2 + 11x + 6}$.

33. If $u = \log \dfrac{(x+2)^2}{\sqrt{(x+1)(x+3)^3}}$, $\dfrac{du}{dx} = \dfrac{x}{(x+1)(x+2)(x+3)}$.

34. If $u = \log \dfrac{2 + x - 2\sqrt{1 + x + x^2}}{x}$, $\dfrac{du}{dx} = \dfrac{1}{x\sqrt{1 + x + x^2}}$.

35. If $u = \log \dfrac{x - 3 + 2\sqrt{2 - 2x}}{1 + x}$, $\dfrac{du}{dx} = \dfrac{1}{1 + x}\sqrt{\dfrac{2}{1 - x}}$.

36. If $u = \log \dfrac{a + bx + 2\sqrt{abx}}{a - bx}$, $\dfrac{du}{dx} = \dfrac{1}{a - bx}\sqrt{\dfrac{ab}{x}}$.

37. If $u = \log \dfrac{\sqrt{a + bx^n} - \sqrt{a}}{\sqrt{bx^n}}$, $\dfrac{du}{dx} = \dfrac{n}{2x}\sqrt{\dfrac{a}{a + bx^n}}$.

38. If $u = \log \dfrac{\sqrt{ax + bx^2} + x\sqrt{b}}{\sqrt{ax + bx^2} - x\sqrt{b}}$, $\dfrac{du}{dx} = \sqrt{\dfrac{b}{ax + bx^2}}$.

39. If $u = \log \dfrac{\sqrt{1 + \sqrt{x}} - 1}{\sqrt{1 + \sqrt{x}} + 1}$, $\dfrac{du}{dx} = \dfrac{1}{2x\sqrt{1 + \sqrt{x}}}$.

40. If $u = \log \dfrac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}$, $\dfrac{du}{dx} = -\dfrac{a}{x\sqrt{a^2 - x^2}}$.

41. If $u = \log \dfrac{\sqrt{1 + x^2} + x\sqrt{2}}{\sqrt{1 - x^2}}$, $\dfrac{du}{dx} = \dfrac{1}{1 - x^2}\sqrt{\dfrac{2}{1 + x^2}}$.

42. If $u = \log \dfrac{3(1 - x) - 2\sqrt{3(1 - x + x^2)}}{1 + x}$, then

$$\frac{du}{dx} = \frac{1}{1 + x}\sqrt{\frac{3}{1 - x + x^2}}.$$

43. If $u = \log \dfrac{\sqrt{1+x^4} + x\sqrt{2}}{1-x^2}$, $\dfrac{du}{dx} = \dfrac{1+a^2}{1-x^2}\sqrt{\dfrac{2}{1+x^4}}$.

44. If $u = \sqrt{x^4 + a^2x^2} + a^2 \log\left(x + \sqrt{x^2+a^2}\right)$, then

$$\frac{du}{dx} = 2\sqrt{x^2+a^2}\,.$$

45. If $u = \dfrac{1}{a+bx^3} - \dfrac{1}{a}\log\left(\dfrac{a+bx^3}{x^3}\right)$, $\dfrac{du}{dx} = \dfrac{3a}{x(a+bx^3)^2}$.

46. If $u = \sqrt{a^2+x^2} + a\log\sqrt{\dfrac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}}$, then

$$\frac{du}{dx} = \frac{\sqrt{a^2+x^2}}{x}.$$

47. If $u = 2\log\dfrac{1+\sqrt{x}}{1-\sqrt{x}} - \dfrac{2}{\sqrt{x}}\log\dfrac{1}{1-x}$, then

$$\frac{du}{dx} = \frac{1}{x^{\frac{3}{2}}}\log\left(\frac{1}{1-x}\right).$$

48. If $u = -\dfrac{a\sqrt{a^2+x^2}}{x^2} + \log\dfrac{a+\sqrt{a^2+x^2}}{x}$, then

$$\frac{du}{dx} = \frac{2a^3}{x^3\sqrt{a^2+x^2}}.$$

49. If $u = 2\sqrt{\dfrac{a}{a+x}} + \log\dfrac{\sqrt{a^2+ax}-a}{\sqrt{a^2+ax}+a}$, then

$$\frac{du}{dx} = \frac{1}{x}\left(\frac{a.}{a+x}\right)^{\frac{3}{2}}.$$

50. If $u = \dfrac{\sqrt{1+x}}{x} + \tfrac{1}{2}\log\dfrac{\sqrt{1+x}-1}{\sqrt{1+x}+1}$, then

$$\frac{du}{dx} = -\frac{1}{x^2\sqrt{1+x}}.$$

51. If $u = \dfrac{3x+1}{x\sqrt{1+x}} + \dfrac{3}{2}\log\dfrac{\sqrt{1+x}-1}{\sqrt{1+x}+1}$, then

$$\frac{du}{dx} = -\frac{1}{x^2(1+x)^{\frac{3}{2}}}.$$

52. If $u = (\log x - 1)\sqrt{a^2+x^2} - \dfrac{a}{2}\log\dfrac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a}$,

$$\frac{du}{dx} = \frac{x\log x}{\sqrt{a^2+x^2}}.$$

53. If $u = \frac{1}{2}\sqrt{\dfrac{a}{2}}\log\dfrac{\sqrt{a+2\sqrt{ax-x^2}}-\sqrt{2a}}{\sqrt{a+2\sqrt{ax-x^2}}+\sqrt{2a}}$

$+\sqrt{a+2\sqrt{ax-x^2}}$, then $\dfrac{du}{dx} = \dfrac{1}{\sqrt{a-x}-\sqrt{x}}$.

54. If $u = \frac{1}{3}(a^2+x^2)^{\frac{3}{2}}\log\dfrac{x^2-b^2}{b^2} + (a^2+b^2)\sqrt{a^2+x^2}$

$$-\frac{2}{9}(a^2+x^2)^{\frac{3}{2}} - (a^2+b^2)^{\frac{3}{2}}\log\frac{a^2+b^2+\sqrt{a^2+b^2}\sqrt{a^2+x^2}}{x^2-b^2},$$

$$\frac{du}{dx} = x\sqrt{a^2+x^2}\log\left(\frac{x^2-b^2}{b^2}\right).$$

55. If $u = \dfrac{1}{\sqrt{-1}}\log(\sqrt{1-x^2}+x\sqrt{-1})$, $\dfrac{du}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.

56. If $\dfrac{x+\sqrt{a^2-u^2}}{a} = \log\dfrac{a+\sqrt{a^2-u^2}}{u}$, then

$$\frac{du}{dx} = -\frac{u}{\sqrt{a^2-u^2}}.$$

CHAP. IV.

MISCELLANEOUS EXAMPLES.

It is required by means of what has been said in Articles (47)—(54), to establish the following results.

1. If $u = \sin mx$, then $\dfrac{du}{dx} = m \cos mx$.

2. If $u = x - \sin x \cos x$, $\dfrac{du}{dx} = 2 \sin^2 x$.

3. If $u = x \sin x + \cos x$, $\dfrac{du}{dx} = x \cos x$.

4. If $u = \tan x - x$, then $\dfrac{du}{dx} = \tan^2 x$.

5. If $u = \cot x - \tan x$, then $\dfrac{du}{dx} = -\dfrac{1}{(\sin x \cos x)^2}$.

6. If $u = \sec^3 x - 3 \sec x$, then $\dfrac{du}{dx} = \dfrac{3 \sin^3 x}{\cos^4 x}$.

7. If $u = \sqrt{\operatorname{cosec} x}$, then $\dfrac{du}{dx} = -\dfrac{\cos x}{2 (\sin x)^{\frac{3}{2}}}$.

8. If $u = 3x + 3 \cot x - \cot^3 x$, $\dfrac{du}{dx} = 3 \cot^4 x$.

9. If $u = 2x \sin x + (2 - x^2) \cos x$, $\dfrac{du}{dx} = x^2 \sin x$.

10. If $u = (2 + \sin^2 x) \cos x$, $\dfrac{du}{dx} = -3 \sin^3 x$.

11. If $u = (2 + 3\cos^2 x)\sin^3 x$, $\dfrac{du}{dx} = 15\sin^2 x\cos^3 x$.

12. If $u = \dfrac{1 - \cos x}{\cos^2 x}$, then $\dfrac{du}{dx} = \dfrac{\sin x\,(2 - \cos x)}{\cos^3 x}$.

13. If $u = \dfrac{2 - 3\cos^2 x}{\sin^3 x}$, $\dfrac{du}{dx} = \dfrac{3\cos^3 x}{\sin^4 x}$.

14. If $u = \dfrac{\cos^3 x - 3\cos x - 3x\sin x}{\sin x}$, $\dfrac{du}{dx} = \dfrac{2\cos^4 x}{\sin^2 x}$.

15. If $u = \dfrac{\sin x}{1 + \tan x}$, $\dfrac{du}{dx} = \dfrac{\cos^3 x - \sin^3 x}{(\cos x + \sin x)^2}$.

16. If $u = e^{\cos x}\sin x$, $\dfrac{du}{dx} = e^{\cos x}(\cos^2 x + \cos x - 1)$.

17. If $u = e^{\sec x}\cot x$, $\dfrac{du}{dx} = e^{\sec x}\left(\dfrac{1 - \cos x - \cos^2 x}{\cos x - \cos^3 x}\right)$.

18. If $u = e^{mx}\cos^m x$, $\dfrac{du}{dx} = m\,e^{mx}\cos^{m-1} x\,(\cos x - \sin x)$.

19. If $u = x^m e^{\sin x}$, $\dfrac{du}{dx} = x^{m-1}e^{\sin x}(m + x\cos x)$.

20. If $u = \log\left(x e^{\cos x}\right)$, $\dfrac{du}{dx} = \dfrac{1}{x} - \sin x$.

21. If $u = e^{ax}(a\sin x - \cos x)$, $\dfrac{du}{dx} = (a^2 + 1)\,e^{ax}\sin x$.

22. If $u = e^{\cos x}\log x$, $\dfrac{du}{dx} = e^{\cos x}\left\{\dfrac{1}{x} - \sin x\log x\right\}$.

23. If $u = \log\sqrt[n]{\dfrac{1 + \sin^n x}{1 - \sin^n x}}$, $\dfrac{du}{dx} = \dfrac{2\sin^{n-1} x\cos x}{1 - \sin^{2n} x}$.

24. If $u = \log\left(\dfrac{1 + \sqrt{-1}\tan x}{1 - \sqrt{-1}\tan x}\right)^m$, $\dfrac{du}{dx} = 2m\sqrt{-1}$.

25. If $u = 2\log\sin x + \operatorname{cosec}^2 x$, $\dfrac{du}{dx} = -2\cot^3 x$.

26. If $u = 2\log\tan x + \sec^2 x$, $\dfrac{du}{dx} = 2\sec^3 x\operatorname{cosec} x$.

27. If $u = 4\log\tan x - \cot 2x$, $\dfrac{du}{dx} = \dfrac{2(4\sin 2x + 1)}{(\sin 2x)^2}$.

28. If $u = \dfrac{2}{\sin^2 x\cos x} - \dfrac{3\cos x}{\sin^2 x} + 3\log\tan\dfrac{x}{2}$, then

$$\frac{du}{dx} = \frac{2}{\sin^3 x\cos^2 x}.$$

29. If $u = \sin^{-1}\dfrac{1-x^2}{1+x^2}$, $\dfrac{du}{dx} = -\dfrac{2}{1+x^2}$.

30. If $u = \cos^{-1}(2x-1)$, $\dfrac{du}{dx} = -\dfrac{1}{\sqrt{x-x^2}}$.

31. If $u = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$, $\dfrac{du}{dx} = \dfrac{1}{1+x^2}$.

32. If $u = \cos^{-1}\dfrac{x^4-1}{x^4+1}$, $\dfrac{du}{dx} = -\dfrac{4x}{x^4+1}$.

33. If $u = \sin^{-1}\sqrt{\dfrac{1-x}{2}}$, $\dfrac{du}{dx} = -\frac{1}{2}\sqrt{\dfrac{1}{1-x^2}}$.

34. If $u = \tan^{-1}\dfrac{2x}{1-x^2}$, $\dfrac{du}{dx} = \dfrac{2}{1+x^2}$.

35. If $u = \cot^{-1}\sqrt{\dfrac{1-x}{x}}$, $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x-x^2}}$.

36. If $u = \tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$, $\dfrac{du}{dx} = \dfrac{1}{2(1+x^2)}$.

37. If $u = \operatorname{cosec}^{-1}\dfrac{1}{2x\sqrt{1-x^2}}$, $\dfrac{du}{dx} = \dfrac{2}{\sqrt{1-x^2}}$.

38. If $u = \dfrac{1}{2\sqrt{ab}}\sin^{-1}\dfrac{2x\sqrt{ab}}{a+bx^2}$, $\dfrac{du}{dx} = \dfrac{1}{a+bx^2}$.

39. If $u = \frac{2}{\sqrt{b}} \tan^{-1} \frac{x\sqrt{b}}{\sqrt{ax - bx^2}}$, $\frac{du}{dx} = \frac{1}{\sqrt{ax - bx^2}}$.

40. If $u = \frac{1}{\sqrt{a}} \cos^{-1} \frac{2a - bx}{bx}$, $\frac{du}{dx} = \frac{1}{x\sqrt{bx - a}}$.

41. If $u = \frac{1}{2\sqrt{a}} \operatorname{vers}^{-1} \frac{2(bx^2 - a)}{bx^2}$, $\frac{du}{dx} = \frac{1}{x\sqrt{bx^2 - a}}$.

42. If $u = \frac{1}{\sqrt{ab}} \cos^{-1} \frac{a - bx}{a + bx}$, $\frac{du}{dx} = \frac{1}{(a + bx)\sqrt{x}}$.

43. If $u = \tan^{-1} \frac{b + 2cx}{\sqrt{4ac - b^2}}$, $\frac{du}{dx} = \frac{\sqrt{4ac - b^2}}{2(a + bx + cx^2)}$.

44. If $u = \sin^{-1} \frac{bx - 2a}{x\sqrt{b^2 + 4ac}}$, $\frac{du}{dx} = \frac{\sqrt{a}}{x\sqrt{cx^2 + bx - a}}$.

45. If $u = \sin^{-1}(3x - 4x^3)$, $\frac{du}{dx} = \frac{3}{\sqrt{1 - x^2}}$.

46. If $u = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, $\frac{du}{dx} = \frac{3}{1 + x^2}$.

47. If $u = \cos^{-1} \sqrt{\frac{1 - x^2}{1 - ax^2}}$, $\frac{du}{dx} = \frac{1}{1 - ax^2} \sqrt{\frac{1 - a}{1 - x^2}}$.

48. If $u = \sin^{-1} \sqrt{\frac{ag + bgx^2}{bf + bgx^2}}$, $\frac{du}{dx} = \frac{x\sqrt{bfg - ag^2}}{(f + gx^2)\sqrt{a + bx^2}}$.

49. If $u = \tan^{-1}\left(\frac{\sqrt{a^2 - b^2} \sin x}{b + a \cos x}\right)$, $\frac{du}{dx} = \frac{\sqrt{a^2 - b^2}}{a + b \cos x}$.

50. If $u = (\sin^{-1} x)^2$, $\frac{du}{dx} = \frac{2 \sin^{-1} x}{\sqrt{1 - x^2}}$.

51. If $u = x - \sqrt{1 - x^2} \sin^{-1} x$, $\frac{du}{dx} = \frac{x \sin^{-1} x}{\sqrt{1 - x^2}}$.

52. If $u = \frac{3x^{\frac{1}{3}}}{\sqrt{1 - x^{\frac{2}{3}}}} - 3 \sin^{-1} x^{\frac{1}{3}}$, $\frac{du}{dx} = \frac{1}{(1 - x^{\frac{2}{3}})^{\frac{3}{2}}}$.

53. If $u = xe^{\tan^{-1}x}$, $\dfrac{du}{dx} = \dfrac{1+x+x^2}{1+x^2}\,e^{\tan^{-1}x}$.

54. If $u = \dfrac{x\sin^{-1}x}{\sqrt{1-x^2}} + \log\sqrt{1-x^2}$, $\dfrac{du}{dx} = \dfrac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}$.

55. If $u = (2x - \tan^{-1}x)\tan^{-1}x - \log(1+x^2)$, then

$$\frac{du}{dx} = \frac{2x^2\tan^{-1}x}{1+x^2}.$$

56. If $u = \dfrac{1}{4\sqrt{2}}\log\dfrac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{x\sqrt{2}}{1-x^2}$,

$$\frac{du}{dx} = \frac{1}{1+x^4}.$$

57. If $u = \dfrac{\sin(l-m+n)x}{l-m+n} + \dfrac{\sin(l+m-n)x}{l+m-n}$

$$-\frac{\sin(l-m-n)x}{l-m-n} - \frac{\sin(l+m+n)x}{l+m+n},\ \frac{du}{dx} = 4\sin mx \sin nx \cos lx.$$

58. If $u = \tan^{-1}\sin^{-1}\dfrac{x}{a}$, $\dfrac{du}{dx} = \dfrac{1}{\sqrt{a^2-x^2}\left\{1+\left(\sin^{-1}\dfrac{x}{a}\right)^2\right\}}$.

59. Given the differential coefficient of $\operatorname{vers} x = \sin x$, it is required to deduce those of $\sin x$, $\cos x$, $\tan x$ and $\sec x$.

60. Given the differential coefficient of $\sec x = \tan x \sec x$, deduce from it the differential coefficients of $\sin x$, $\cos x$, &c.

61. From the differential coefficient of each of the Trigonometrical functions, it is required to deduce those of all the rest.

62. Having obtained the differential of the arc of a circle, infer those of the sine, cosine, tangent, &c. of the arc.

CHAP. V.

MISCELLANEOUS EXAMPLES.

I. *Successive Differentiations as in* (65) *and* (66).

1. If $u = ax^3 - bx^2 + cx - e$, then

$$\frac{d^2u}{dx^2} = 6ax - 2b \text{ and } \frac{d^3u}{dx^3} = 6a.$$

2. If $u = mx^{m+1} - (m+1)\,x^m + 1$, then

$$\frac{d^2u}{dx^2} = m\,(m+1)\,\{mx^{m-1} - (m-1)\,x^{m-2}\}.$$

3. If $u = x^4 + 2ax^3 + 2a^2x^2 + 2a^3x + a^4$, then

$$\frac{d^3u}{dx^3} = 12\,(2x + a) \text{ and } \frac{d^4u}{dx^4} = 24.$$

4. If $u = xe^x$, then $\frac{d^2u}{dx^2} = (x+2)\,e^x$,

$$\frac{d^3u}{dx^3} = (x+3)\,e^x,\ \frac{d^4u}{dx^4} = (x+4)\,e^x, \text{ \&c.}$$

5. If $u = x^2 \log x$, then $\frac{d^2u}{dx^2} = 3 + 2\log x$,

$$\frac{d^3u}{dx^3} = 2x^{-1},\ \frac{d^4u}{dx^4} = -1\,.\,2x^{-2}, \text{ \&c.}$$

6. If $u = 1 - 2\sin^2\frac{x}{2}$, then $\frac{d^2u}{dx^2} = -\cos x$,

$$\frac{d^3u}{dx^3} = \sin x,\ \frac{d^4u}{dx^4} = \cos x = u, \text{ \&c.}$$

7. If $u = x\sin x$, then $\frac{d^3u}{dx^3} = -3\sin x - x\cos x$.

8. If $u = e^x \cos x$, then $\dfrac{d^2u}{dx^2} = -2e^x \sin x$,

$$\frac{d^3u}{dx^3} = -2e^x(\sin x + \cos x),\quad \frac{d^4u}{dx^4} = -4e^x \cos x = -4u, \text{ \&c.}$$

9. If $u = \sec x$, $\dfrac{d^3u}{dx^3} = \sec x \tan x\,(6\sec^2 x - 1)$.

10. If $u = x \tan x$, $\dfrac{d^2u}{dx^2} = 2\sec^2 x\,(1 + x\tan x)$.

11. If $u = e^x \cot x$, $\dfrac{d^2u}{dx^2} = e^x(2\cot^3 x - 2\cot^2 x + 3\cot x - 2)$.

12. If $u = \sin^{-1} x$, $\dfrac{d^5u}{dx^5} = 9(1 - x^2)^{-\frac{5}{2}}$

$$+ 2\,.\,5\,.\,9x^2(1-x^2)^{-\frac{7}{2}} + 3\,.\,5\,.\,7x^4(1-x^2)^{-\frac{9}{2}}.$$

13. If $u = \tan^{-1} x$, $\dfrac{d^4u}{dx^4} = \dfrac{3\,.\,2^3x}{(1+x^2)^3} - \dfrac{3\,.\,2^4x^3}{(1+x^2)^4}$.

14. If $u = (1+x^2)\tan^{-1} x$, $\dfrac{d^3u}{dx^3} = \dfrac{4}{(1+x^2)^2}$.

15. If $u = \sec^{-1} x$, $\dfrac{d^2u}{dx^2} = -\dfrac{1}{x^2\sqrt{x^2-1}} - \dfrac{1}{(x^2-1)^{\frac{3}{2}}}$.

16. If $u = \dfrac{1}{1+x^2}$, $\dfrac{d^4u}{dx^4} = \dfrac{24 - 240x^2 + 120x^4}{(1+x^2)^5}$.

17. If $u = \dfrac{1}{x}$, $\dfrac{d^nu}{dx^n} = (-1)^n \dfrac{1\,.\,2\,.\,3\,.\,\text{\&c.}\,n}{x^{n+1}}$.

18. If $u^2 - 2ux + a^2 = 0$, $\dfrac{d^2u}{dx^2} = \dfrac{u(u-2x)}{(u-x)^3}$.

19. If $au^3 - xu - a = 0$, $\dfrac{d^2u}{dx^2} = -\dfrac{2ux}{(3au^2 - x)^3}$.

20. If $u = x(a-x)^m$, find the value of $\dfrac{d^nu}{dx^n}$.

21. If $u = x^m a^x$, find the value of $\dfrac{d^nu}{dx^n}$.

22. If $u = x^m \log x$, find the value of $\dfrac{d^n u}{dx^n}$.

23. If $u = x^m \sin x$, find the value of $\dfrac{d^n u}{dx^n}$.

24. If $u = e^x \cos x$, find the value of $\dfrac{d^n u}{dx^n}$.

25. If $u = e^{mx} \sin x$, it is required to prove that

$$\frac{d^n u}{dx^n} = \frac{e^{mx} \sin (x + n\phi)}{\sin^n \phi}, \text{ where } \phi = \cot^{-1} m.$$

For the solution of the last six examples see Ex. 6. of Article (66).

II. *Elimination of constant Quantities and irrational or transcendental Functions as in* (67) *and* (68).

1. If $u + ax + b = 0$, the two derivatives of the first order are $u - x\dfrac{du}{dx} + b = 0$ and $\dfrac{du}{dx} + a = 0$; and that of the second order is $x\dfrac{d^2u}{dx^2} = 0$.

2. If $x^2 - 2au + a^2 = b^2$, the two derivatives of the first order are $x - a\dfrac{du}{dx} = 0$ and $x + u\dfrac{du}{dx} = \sqrt{u^2 - x^2 + b^2}\,\dfrac{du}{dx}$; and the derivative of the second order is $\dfrac{du}{dx} - x\dfrac{d^2u}{dx^2} = 0$.

3. If c be eliminated from the equation $x^2 + y^2 = cx$, then $y - px = x\sqrt{1 + p^2}$, if $p = \dfrac{dy}{dx}$.

4. If $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = a^2$, it is required to prove that $\dfrac{1}{\sqrt{1 - x^2}} + \dfrac{p}{\sqrt{1 - y^2}} = 0$.

5. If $\sqrt{1-x^2}+\sqrt{1-y^2}=c\,(x-y)$, then will

$$\frac{1}{\sqrt{1-x^2}}=\frac{p}{\sqrt{1-y^2}}.$$

6. If $(x-y)\,(2a+x+y)^3=b^4$, then will $(a+x+2y)p$ $=a+2x-y$ by the elimination of b.

7. If $(y-x)^{m-1}\,(y+x)^{m+1}=a^{2m}$, then will

$$\frac{1}{x}-\frac{p}{y}=m\left(\frac{1}{y}-\frac{p}{x}\right).$$

8. If $y=ax+c\sqrt{1+x^2}$, then will $(1+x^2)\,p=a+xy$.

9. If $y^2=a\,(x+\sqrt{1+x^2})^2-\frac{2}{3}b\,(x+2\sqrt{1+x^2})$,

then will $py=a+\dfrac{y^2}{\sqrt{1+x^2}}$.

10. If $xe^{\frac{x}{y}}=c$, then will $xy+y^2=px^2$.

11. If $c\,(x+y)=e^{y-x}$, then $x+y+1=p\,(x+y-1)$.

12. If $(x+y)\log\dfrac{x}{c}=xe^{\frac{y}{x}}$, then will

$$pxy=y^2+(x+y)^2e^{-\frac{y}{x}}.$$

13. If $\log\dfrac{x+y}{a}+\dfrac{x}{x+y}=0$, then will $py+x+2y=0$.

14. If $\log\dfrac{x}{a}=\sin^{-1}\dfrac{y}{x}$, then will $px-y=\sqrt{x^2-y^2}$.

15. If $\tan^{-1}\dfrac{y}{x}=a\log\dfrac{\sqrt{x^2+y^2}}{b}$, then will

$$(x-ay)\,p=ax+y.$$

16. If $ay^2+2xy-bx^2=0$, then will

$$(xy-bx^2)\,p+bxy-y^2=0,\quad xy+ay^2-(x^2+axy)\,p=0$$

and $(px-y)^3=0$.

17. If $y=xe^{\frac{x}{a}}$, then will $px=y\left(1+\log\dfrac{y}{x}\right)$.

18. If $\frac{1}{y} = ae^{2x} + \frac{1}{4}(2x^2 + 2x + 1)$, then will $p + 2y = x^2y^2$.

19. If $x^2y - x = a \tan \frac{a + bx}{x}$, then will $p + y^2 + \frac{a^2}{x^4} = 0$.

20. If $y = xe^{\tan^{-1} x}$, then will $px(1 + x^2) = (1 + x + x^2)$.

21. If $y = \frac{e^{2x} + 1}{e^{2x} - 1}$, then will $p = 1 - y^2$.

22. If $y = a \sin(x + b)$, then $q + y = 0$, if $q = \frac{d^2y}{dx^2}$.

23. If $y = ae^x + be^{-x}$, then will $q - y = 0$.

24. If $2y = ae^x + be^{-x} + c$, then will $q = \sqrt{ab + p^2}$.

25. If $y = \log \sin(ax + b)$, then will $q + p^2 + a^2 = 0$.

26. If $ae^y = x^2 - b^2$, then will $p - qx = p^2x$.

27. If $y^2 + ax + b = 2cx \log x$, then $qxy + p^2x = c$.

28. If $y = mx \log \frac{max}{b(1 - ax)}$, then $mqx^3 = (y - px)^2$.

29. If $y = \log(e^x + e^{-x})$, then will $2x = \log\left(\frac{1 + p}{1 - p}\right)$,
and $p^2 + q = 1$.

30. If $y = e^{\sec^{-1} x}$, then will $y = (1 - x^2)q - px$.

31. If $y = e^x \cos x$, then will $q - 2p + 2y = 0$.

32. If $y = e^x \sec x$, then will $(p + y)qy = p^3 + y^3$.

33. If $\frac{1}{x} = \frac{p}{\sqrt{2(y^4 - a^4)}}$, then will $qx^2 + px = 4y^3$.

34. If $(1 - x)y = ax^2 + bx + c$, then will
$(1 - x)r - 3q = 0$, if $r = \frac{d^3y}{dx^3}$.

35. If $y = e^x(a + bx + cx^2)$, then $r - 3q + 3p - y = 0$.

36. If $y = ae^{x} + be^{-x} + c \sin (x + e)$, then will $s - y = 0$, if $s = \dfrac{d^4 y}{dx^4}$.

37. If $y = e^{x} (a + bx) + c \sin (x + e)$, then will

$$s - 2r + 2q - 2p + y = 0.$$

38. If $y = ae^{\alpha x} + be^{\beta x} + ce^{\gamma x} +$ &c. to n terms, where α, β, γ, &c. are the n roots of 1, then will $\dfrac{d^n y}{dx^n} - y = 0$.

39. If $yx + (1 - c)(y + x) = c$, it is required to eliminate c.

40. Eliminate a and b from the equation

$$ax^2 + by^2 = ax + by.$$

41. Eliminate $\sin x$ and $\sin 2x$ from the equation

$$y = a \sin x + b \sin 2x.$$

42. From $y = ae^{kx} + be^{lx} - \dfrac{c}{mn} \cos nx$, it is required to eliminate the terms involving a and b, when k and l are the roots of the equation $z^2 + mz + n^2 = 0$.

43. Eliminate the constant quantities and exponential function from the equation $y = e^{x} (a + bx + cx^2 +$ &c. $+ lx^{n-1})$.

III. *Developement of Functions, &c. as in Articles* (69)–(86).

1. If $u = (a - x)^4$, then $u = a^4 - 4a^3 x + 6a^2 x^2 - 4ax^3 + x^4$.

2. If $u = (a^2 + x^2)^5$, then will

$$u = a^{10} + 5a^8 x^2 + 10a^6 x^4 + 10a^4 x^6 + 5a^2 x^8 + x^{10}.$$

3. If $u = \sqrt{1 + x}$, then will

$$u = 1 + \frac{1}{2} x - \frac{1}{2^2} \frac{x^2}{1.2} + \frac{3}{2^3} \frac{x^3}{1.2.3} - \frac{3.5}{2^4} \frac{x^4}{1.2.3.4} + \text{\&c.}$$

4. If $u = a^2 (1 - x^2)^{-\frac{1}{5}}$, then will

$$u = a^2 + \frac{a^2 x^2}{5} + \frac{6a^2 x^4}{5.10} + \frac{6.11 a^2 x^6}{5.10.15} + \frac{6.11.16 a^2 x^8}{5.10.15.20} + \text{\&c.}$$

5. If $u=(a+x)^{-4}$, then will

$$u=\frac{1}{a^4}-\frac{4x}{a^5}+\frac{10x^2}{a^6}-\frac{20x^3}{a^7}+\frac{35x^4}{a^8}-\&c.$$

6. If $u=(a+x)^{\frac{1}{3}}$, then will $u=a^{\frac{1}{3}}$ multiplied into

$$\left\{1+\frac{1}{3}\frac{x}{a}-\frac{1.2}{3^2}\frac{x^2}{1.2a^2}+\frac{1.2.5}{3^3}\frac{x^3}{1.2.3a^3}-\frac{1.2.5.8}{3^4}\frac{x^4}{1.2.3.4a^4}+\&c.\right\}$$

7. If $u=e^{\pm ax}$, then $u=1\pm\frac{ax}{1}+\frac{a^2x^2}{1.2}\pm\frac{a^3x^3}{1.2.3}+\&c.$

8. If $u=\log(a+bx)$, then will

$$u=\log a+\frac{b}{a}\frac{x}{1}-\frac{b^2}{a^2}\frac{x^2}{2}+\frac{b^3}{a^3}\frac{x^3}{3}-\frac{b^4}{a^4}\frac{x^4}{4}+\&c.$$

9. If $u=\log\left(\frac{a+bx}{a-bx}\right)$, then will

$$u=2\left(\frac{b}{a}x+\frac{b^3}{a^3}\frac{x^3}{3}+\frac{b^5}{a^5}\frac{x^5}{5}+\frac{b^7}{a^7}\frac{x^7}{7}+\&c.\right).$$

10. If $u=\sin(a+bx)$, then will

$$u=\sin a+b\cos a\frac{x}{1}-b^2\sin a\frac{x^2}{1.2}-b^3\cos a\frac{x^3}{1.2.3}+\&c.$$

11. If $u=\cos(ax-b)$, then will

$$u=\cos b+a\sin b\frac{x}{1}-a^2\cos b\frac{x^2}{1.2}-a^3\sin b\frac{x^3}{1.2.3}+\&c.$$

12. If $u=\sin^{-1}\frac{x}{a}$, then will $u=\frac{x}{a}+\frac{x^3}{1.2a^3}+\&c.$

13. If $u=\tan x$, then $u=\frac{x}{1}+\frac{2x^3}{1.2.3}+\frac{2^4x^5}{1.2.3.4.5}+\&c.$

14. If $u=\tan^{-1}\frac{x}{a}$, then will

$$u=\frac{x}{1.a}-\frac{x^3}{3.a^3}+\frac{x^5}{5.a^5}-\frac{x^7}{7.a^7}+\&c.$$

15. If $u=e^{\sin x}$, then $u=1+\frac{x}{1}+\frac{x^2}{1.2}-\frac{x^4}{2.4}-\&c.$

16. If $u=\cos^m x$, then $u=1-\frac{mx^2}{1.2}+(3m-2)\,m\frac{x^4}{1.2.3.4}-$&c.

17. If $u=\sec x$, then will

$$u=1+\frac{x^2}{1.2}+\frac{5x^4}{1.2.3.4}+\frac{61x^6}{1.2.3.4.5.6}+\&c.$$

18. If $u=\log\sec x$, then will

$$u=\frac{x^2}{1.2}+\frac{2x^4}{1.2.3.4}+\frac{16x^6}{1.2.3.4.5.6}+\&c.$$

19. If $u=e^{\cos x}$, then will $u=e-\frac{ex^2}{1.2}+\frac{ex^4}{1.2.3}-$&c.

20. If $u=e^x\sec x$, then $u=1+x+\frac{2x^2}{1.2}+\frac{4x^3}{1.2.3}+$&c.

21. If $u=a^{\sin^{-1}x}$, then will

$$u=1+\log a\frac{x}{1}+(\log a)^2\frac{x^2}{1.2}+\left\{(\log a)^3+\log a\right\}\frac{x^3}{1.2.3}$$

$$+\left\{(\log a)^4+4(\log a)^2\right\}\frac{x^4}{1.2.3.4}+\&c.$$

22. Expand $\frac{1}{1-x}$ to six terms by means of *Maclaurin's* Theorem.

23. Develope $\tan(a+bx)$ in a series of the form $A+Bx+Cx^2+$&c. by *Maclaurin's* Theorem.

24. Expand $\log\cot(45^0-\frac{1}{2}x)$ in a series ascending by powers of x.

25. Expand $e^{\tan x}$ to four terms by means of *Maclaurin's* Theorem.

26. Develope $e^{\cos\frac{1}{2}(\pi+x)}$ by *Maclaurin's* Theorem up to the term involving x^4.

27. If $\sin y=m\sin x$, express y in a series of the form $Ax+Bx^2+Cx^3+$&c.

28. If $\tan y=m\tan x$, expand y in a series of the form $A+Bx+Cx^2+$&c.

29. By *Maclaurin's* Theorem expand $\sin(a+bx+cx^2+\&c.)$ into a series of the form $A+Bx+Cx^2+\&c.$

30. Having given $1+x\sin y=\tan y$, find y in terms of x by *Maclaurin's* Theorem.

31. If $x=m\tan(z-nx)$, prove by *Maclaurin's* Theorem that when $\frac{x}{z}$ is very small, $x=\frac{m}{2}\,\frac{\sin 2z}{mn+\cos^2 z}$ very nearly.

32. Expand $e^{a+bx+cx^2+\&c.}$ by *Maclaurin's* Theorem.

33. Prove the following formula:

$$\cot x=x^{-1}-\frac{x}{3}-\frac{x^3}{3^2.5}-\frac{2x^5}{3^2.5.7}-\frac{x^7}{3^2.5^2.7}-\&c.$$

34. Shew that $\operatorname{cosec} x=\frac{1}{x}+\frac{x}{1.2.3}+\frac{14x^3}{1.2.3.4.5.6}+\&c.$

35. By *Maclaurin's* Theorem find the developement of $\frac{x}{x-1}$ in descending powers of x.

36. Apply *Maclaurin's* Theorem to express the arc of a circle in terms of the secant.

37. If $my^3-xy=m$, then $y=1+\frac{x}{3m}-\frac{x^3}{3^4m^3}+\frac{x^4}{3^5m^4}-\&c.$

38. If $my^3-x^3y-mx^3=0$, then will

$$y=-m-\frac{m^4}{x^3}-\frac{3m^7}{x^6}-\frac{12m^{10}}{x^9}-\&c.$$

39. If $y^3-2xy^2+x^2y-a^3=0$, then will

$$y=\frac{a^3}{x^2}+\frac{2a^6}{x^5}+\frac{7a^9}{x^8}+\&c.$$

40. If $y^3-3y+x=0$, then $y=\frac{x}{3}+\frac{x^3}{3^4}+\frac{x^5}{3^6}+\&c.,$

$$\text{also, } y=\pm\sqrt{3}-\frac{1}{6}x\mp\frac{\sqrt{3}}{81}x^2-\&c.$$

$$\text{and } y=-x^{\frac{1}{3}}-\frac{1}{x^{\frac{1}{3}}}-\frac{2}{1.2x}+\frac{4}{1.2.3x^{\frac{4}{3}}}-\&c.$$

41. If $y^3+a^2y-2a^3+axy-x^3=0$, then will

$$y=a-\frac{x}{2^2}+\frac{x^2}{2^6a}-\&c. \text{ and } y=x+\frac{a}{3}+\frac{2a^2}{9x}+\&c.$$

42. If $y^3-a^2y+axy-x^3=0$, then will

$$y=x-\frac{a}{3}+\frac{a^2}{3x}+\frac{a^3}{81x^2}+\&c.,\ y=\pm a\mp\frac{x}{2}\mp\frac{x^2}{8a}+\&c.$$

$$\text{and } y=-\frac{x^3}{a^2}-\frac{x^4}{a^3}-\frac{x^5}{a^4}-\frac{x^6}{a^5}-\&c.$$

43. It is required to prove that

$$\log\left(\frac{x}{x-1}\right)=\frac{1}{x-1}-\frac{1}{2}\,\frac{1}{(x-1)^2}+\frac{1}{3}\,\frac{1}{(x-1)^3}-\&c.$$

44. Find the developement of $u=e^{e^x}$ to five terms.

45. Given the differential coefficients of $\frac{1}{1+x}$, it is required to deduce the differential coefficients of $\log\frac{1}{1+x}$.

46. From the differential coefficients of $\cos x$, it is required to derive those of $\log\cos x$.

47. Given the differential coefficients of $\log x$, to deduce those of log of $\log x$, or of $\log^2 x$.

48. If $u=f(x)$, it is required to prove that

$$f(2x)=f(x)+\frac{x\,du}{dx}+\frac{x^2d^2u}{1.2\,dx^2}+\frac{x^3d^3u}{1.2.3\,dx^3}+\&c.:$$

$$f(0)=f(x)-\frac{x\,du}{dx}+\frac{x^2d^2u}{1.2\,dx^2}-\frac{x^3d^3u}{1.2.3\,dx^3}+\&c.$$

49. On the same hypothesis will there result

$$f\left(\frac{3x}{2}\right)=f(x)+\frac{du}{dx}\,\frac{x}{2}+\frac{d^2u}{dx^2}\,\frac{x^2}{1.2.2^2}+\&c.:$$

$$f\left(\frac{x}{2}\right)=f(x)-\frac{du}{dx}\,\frac{x}{2}+\frac{d^2u}{dx^2}\,\frac{x^2}{1.2.2^2}-\&c.$$

50. On the same supposition will $f(mx) = f(x)$

$$+ (m-1)\frac{x\,du}{dx} + (m-1)^2 \frac{x^2 d^2u}{1\,.\,2\,dx^2} + (m-1)^3 \frac{x^3 d^3u}{1\,.\,2\,.\,3\,dx^3} + \&c.:$$

$$f\left(\frac{x^2+2x}{1+x}\right) = f(x) + \frac{du}{dx}\,\frac{x}{1+x} + \frac{d^2u}{dx^2}\,\frac{x^2}{1\,.\,2\,(1+x)^2}$$

$$+ \frac{d^3u}{dx^3}\,\frac{x^3}{1\,.\,2\,.\,3\,(1+x)^3} + \&c.$$

$$\text{and } f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{du}{dx}\,\frac{x}{1+x} + \frac{d^2u}{1\,.\,2\,dx^2}\left(\frac{x}{1+x}\right)^2$$

$$- \frac{d^3u}{1\,.\,2\,.\,3\,dx^3}\left(\frac{x}{1+x}\right)^3 + \&c.$$

51. If $u = x^m$, then will

$$(x+nx)^m \text{ or } (1+n)^m x^m = x^m + m x^{m-1}\,nx + \frac{m(m-1)}{1\,.\,2} x^{m-2} n^2 x^2$$

$$+ \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3} x^{m-3} n^3 x^3 + \&c.$$

52. If $u = \log x$, then will $\log 2x = \log x + 1 - \frac{1}{2} + \frac{1}{3} - \&c.:$ and $\log mx = \log x + (m-1) - \frac{1}{2}(m-1)^2 + \frac{1}{3}(m-1)^3 - \&c.$

53. If $u = \sin x$, then will

$$\sin 2x = \sin x + x\cos x - \frac{x^2 \sin x}{1\,.\,2} - \frac{x^3 \cos x}{1\,.\,2\,.\,3} + \&c.$$

and $\sin mx$

$$= \sin x + (m-1)\,x\cos x - \frac{(m-1)^2 x^2}{1\,.\,2}\sin x - \frac{(m-1)^3 x^3}{1\,.\,2\,.\,3}\cos x + \&c.$$

54. It is required to prove that

$$(e^m \pm x)^{\frac{1}{m}} = e\left\{1 \pm \frac{1}{m}\left(\frac{x}{e^m \pm x}\right) + \frac{(m+1)}{1\,.\,2\,m^2}\left(\frac{x}{e^m \pm x}\right)^2\right.$$

$$\left.\pm \frac{(m+1)(2m+1)}{1\,.\,2\,.\,3\,m^3}\left(\frac{x}{e^m \pm x}\right)^3 + \&c.\right\}.$$

55. If $u = f(x)$, it is required to shew that

$$f(x) = f(x+h) - \frac{df(x+2h)}{dx}\,\frac{2h}{1\,.\,2} + \frac{d^2 f(x+3h)}{dx^2}\,\frac{3^2h^2}{1\,.\,2\,.\,3} - \&c.$$

56. It is required to prove by *Taylor's* Theorem, that

$$\tan(x+h) = \tan x + \sec^2 x \frac{h}{1} + 2\sin x \sec^3 x \frac{h^2}{1.2} + \&c.$$

57. Shew by the same Theorem, that

$$\sin^{-1}(x+h) = \sin^{-1}x + \frac{1}{(1-x^2)^{\frac{1}{2}}}\frac{h}{1} + \frac{x}{(1-x^2)^{\frac{3}{2}}}\frac{h^2}{1.2}$$

$$+ \frac{2x^2+1}{(1-x^2)^{\frac{5}{2}}}\frac{h^3}{1.2.3} + \frac{3x(2x^2+3)}{(1-x^2)^{\frac{7}{2}}}\frac{h^4}{1.2.3.4} + \&c.$$

58. By the same means it is required to prove that

$$\tan^{-1}(x+h) = \tan^{-1}x + \frac{1}{1+x^2}\frac{h}{1} - \frac{2x}{(1+x^2)^2}\frac{h^2}{1.2}$$

$$+ \frac{2(3x^2-1)}{(1+x^2)^3}\frac{h^3}{1.2.3} - \frac{2.3.4x(x^2-1)}{(1+x^2)^4}\frac{h^4}{1.2.3.4} + \&c.$$

59. Given $u = (a - bx)^m$, to find the value of $\frac{d^n u}{dx^n}$.

60. Given $u = (1 - x^2)^m$, to find the value of $\frac{d^n u}{dx^n}$.

61. Given $u = (2x + x^2)^m$, to find the value of $\frac{d^n u}{dx^n}$.

62. Given $u = (1 + 2x + 3x^2)^5$, to find the value of $\frac{d^n u}{dx^n}$, and to determine the greatest value n admits of.

63. Given $u = \frac{1}{(a - bx)^m}$, to find the value of $\frac{d^n u}{dx^n}$.

64. Given $u = \frac{1}{(a^2 + x^2)^m}$, to find the value of $\frac{d^n u}{dx^n}$.

65. Given $u = \frac{1}{(2x - x^2)^m}$, to find the value of $\frac{d^n u}{dx^n}$.

66. Given $u = \frac{1}{(1 - 2x + 3x^2)^{\frac{1}{2}}}$, to find the value of $\frac{d^n u}{dx^n}$.

67. Given $u = \frac{1}{(a + bx + cx^2)^{\frac{3}{2}}}$, to find the value of $\frac{d^5u}{dx^5}$.

For the solution of the last nine Examples, the reader is referred to Ex. (2) of Article (82).

68. If $(y^2 - ax)\frac{dy}{dx} - ay + x^2 = 0$, where x is the independent variable, then will $(x^2 - ay)\frac{dx}{dy} - ax + y^2 = 0$, where y is the independent variable.

69. If $\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3 = x\frac{d^2y}{dx^2}$, where x is the independent variable, then will $x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 = 1$, where y is the independent variable.

70. If $y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = y$, where x is the independent variable, then will $y\frac{d^2x}{dy^2} + y\left(\frac{dx}{dy}\right)^3 = 2\frac{dx}{dy}$, where y is the independent variable.

71. If $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 = 0$, where x is the independent variable, then will $x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 = x\frac{dx}{dy}$, where y is the independent variable.

72. If $(x + a)\frac{d^2y}{dx^2} + \frac{x}{b}\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx} = 0$, where x is the independent variable, then will

$$(x + a)\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 - \frac{x}{b}\left(\frac{dx}{dy}\right) = 0,$$

where y is the independent variable.

73. If $(dy^2 + dx^2)^{\frac{3}{2}} + a\,dx\,d^2y = 0$, where x is the independent variable, then will $(dy^2 + dx^2)^{\frac{3}{2}} - a\,dy\,d^2x = 0$, where y is the independent variable.

74. If $\frac{d^2x}{dy^2} + \frac{1}{x}\left(\frac{dx}{dy}\right)^2 - \frac{dx}{dy} = 0$, where y is the independent variable, then will $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 = 0$, where x is the independent variable.

75. If x be the independent variable in

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0,$$

find the equation when y is the independent variable.

76. Transform $a\frac{d^3y}{dx^3} - bx\frac{d^2y}{dx^2} + cx^2\frac{dy}{dx} - x^3y = 0$, where x is the principal variable, into a formula where y shall be the principal variable.

77. If x be the principal variable in

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0,$$

find the corresponding equation when y is the independent variable.

The last ten Examples are dependent upon the principles explained in Article (83).

78. If $x\frac{dy}{dx} - ay = 0$, where x is the independent variable, then will $\frac{dy}{dz} - ay = 0$, where $z = \log x$ is the independent variable.

79. If θ be the independent variable in

$$\frac{d^2y}{d\theta^2} + 2y\frac{dy}{d\theta} + \sec^2\theta = 0,$$

then will $(1 + x^2)\frac{d^2y}{dx^2} + 2(2x + y)\frac{dy}{dx} + 1 = 0$, where $x = \tan\theta$ is the principal variable.

80. Change $\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0$, where x is the independent variable, so that $\theta = \cos^{-1}x$ shall be the independent variable.

81. If $\frac{d^4y}{dx^4} - \frac{4}{x}\frac{d^3y}{dx^3} + \frac{12}{x^2}\frac{d^2y}{dx^2} - \frac{24}{x^3}\frac{dy}{dx} + \frac{24}{x^4}y = 0$, where x is the independent variable, find the equation wherein $z = \log x$ is the independent variable.

82. Reduce the equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where dx is considered constant, into an equation wherein $\sqrt{dx^2 + dy^2}$ is constant.

83. If $a\sqrt{(d^2x)^2 + (d^2y)^2} = ds^2$, where $ds = \sqrt{dx^2 + dy^2}$ is invariable, it is required to find the corresponding formulæ when x and y are respectively the principal variables.

84. If ydx be invariable, and for $\frac{d^2x}{dx^2}$, $\frac{d^3x}{dx^3}$, &c., there be substituted $-\frac{p}{y}$, $\frac{3p^2 - qy}{y^2}$, &c., a homogeneous function of an order above the second, wherein neither x nor y is the independent variable, will become a function of y and its differential coefficients relatively to x: required a proof.

85. Apply the formula $\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$ to find the differential coefficients of $u = (a^x)^{\cos x}$ and $a^{\left(x^{\cos x}\right)}$.

86. From the same formula, if $y = a\,(2\cos\theta - \cos 2\theta)$ and $x = a\,(2\sin\theta - \sin 2\theta)$, deduce the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

87. If $x = f(t)$ and $y = \phi(t)$, it is required to express $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. in terms of t.

The last ten Examples are illustrative of the principles explained in Articles (84) and (85).

CHAP. VI.

MISCELLANEOUS EXAMPLES.

1. If $u = \dfrac{x^3 - 1}{x^3 - 2x^2 + 2x - 1}$; when $x = 1$, $u = 3$.

2. If $u = \dfrac{x^3 - 19x + 30}{x^3 - 2x^2 - 9x + 18}$; when $x = 2$, $u = \dfrac{7}{5}$, and when $x = 3$, $u = \dfrac{4}{3}$.

3. If $u = \dfrac{1 + x - x^2 - x^3}{1 + 2x + 2x^2 + 2x^3 + x^4}$; when $x = -1$, $u = 1$.

4. If $u = \dfrac{x^3 - 5x^2 + 8x - 6}{x^3 - 7x^2 + 16x - 12}$; when $x = 3$, $u = 5$.

5. If $u = \dfrac{1 - 6x^2 + 5x^4}{1 - 4x^2 - 5x^4}$; when $x = \pm \dfrac{1}{\sqrt{5}}$, $u = \dfrac{2}{3}$.

6. If $u = \dfrac{x^4 - 4x^3 + 8x^2 - 16x + 16}{x^4 - 6x^3 + 13x^2 - 12x + 4}$; when $x = 2$, $u = 8$.

7. If $u = \dfrac{x^4 + 3x^3 - 7x^2 - 27x - 18}{x^4 - 3x^3 - 7x^2 + 27x - 18}$; when $x = 3$, $u = 10$, and when $x = -3$, $u = \dfrac{1}{10}$.

8. If $u = \dfrac{x^5 + ax^4 - a^4x - a^5}{x^4 + 2ax^3 + 2a^2x^2 + 2a^3x + a^4}$;
when $x = -a$, $u = -2a$.

9. If $u = \dfrac{a(x^2 + c^2) - 2acx}{b(x^2 + c^2) - 2bcx}$; when $x = c$, $u = \dfrac{a}{b}$.

10. If $u=\dfrac{2}{x^2-1}-\dfrac{1}{x-1}$; when $x=1$, $u=-\dfrac{1}{2}$.

11. If $u=\dfrac{a-\sqrt[n]{a^n-x^n}}{x^n}$; when $x=0$, $u=\dfrac{1}{n a^{n-1}}$.

12. If $u=\dfrac{\sqrt{a+x}-\sqrt{2a}}{\sqrt{a+2x}-\sqrt{3a}}$; when $x=a$, $u=\frac{1}{4}\sqrt{6}$.

13. If $u=\dfrac{xe^{2x}+1-e^{2x}-x}{e^{2x}-1}$; when $x=0$, $u=-1$.

14. If $u=\dfrac{xe^{2x}+xe^{x}-2e^{2x}+2e^{x}}{(e^{x}-1)^3}$; when $x=0$, $u=\dfrac{1}{6}$.

15. If $u=\dfrac{a^n-x^n}{\log(a^n)-\log(x^n)}$; when $x=a$, $u=a^n$.

16. If $u=\dfrac{x}{x-1}-\dfrac{1}{\log x}$; when $x=1$, $u=\frac{1}{2}$.

17. If $u=\dfrac{e^{2x}-1}{e^{x}\log(1+x)}$; when $x=0$, $u=2$.

18. If $u=\dfrac{e^{x}-1-\log(1+x)}{x^2}$; when $x=0$, $u=1$.

19. If $u=\dfrac{a^{\log x}-x}{\log x}$; when $x=1$, $u=\log a-1$.

20. If $u=(1-x)\log(1-x)$; when $x=1$, $u=0$.

21. If $u=\dfrac{\cos x-\cos 2x}{\cos x-\cos 3x}$; when $x=0$, $u=\frac{3}{8}$.

22. If $u=\dfrac{2(\sin x)^2+\sin x-1}{2(\sin x)^2-3\sin x+1}$; when $x=30^\circ$, $u=-3$.

23. If $u=\dfrac{\tan x+\sec x-1}{\tan x-\sec x+1}$; when $x=0$, $u=1$.

24. If $u = \dfrac{\sin 2x + 2(\sin x)^2 - 2\sin x}{1 - \cos x}$; when $x = 0$, $u = 4$.

25. If $u = \dfrac{\tan x - \sin x}{(\sin x)^3}$; when $x = 0$, $u = \frac{1}{2}$.

26. If $u = \dfrac{1 + \cos 2\pi x}{e^{2x} - 2ex}$; when $x = \frac{1}{2}$, $u = \dfrac{\pi^2}{e}$.

27. If $u = \dfrac{e^x - e^{\sin x}}{x - \sin x}$; when $x = 0$, $u = 1$.

28. If $u = \dfrac{(e^x - e^{-x})^2}{x^2 \cos x}$; when $x = 0$, $u = 2$.

29. If $u = \dfrac{1}{\sin^2 x} - \dfrac{1}{x^2}$; when $x = 0$, $u = \frac{1}{3}$.

30. If $u = \dfrac{1}{2x^2} - \dfrac{1}{2x \tan x}$; when $x = 0$, $u = \dfrac{1}{6}$.

31. If $u = \dfrac{x^2 - a^2}{x^2} \tan \dfrac{\pi x}{2a}$; when $x = a$, $u = -\dfrac{4}{\pi}$.

32. If $u = \sec \dfrac{\pi x}{2} \log \dfrac{1}{x}$; when $x = 1$, $u = \dfrac{2}{\pi}$.

33. If $u = \dfrac{\tan(a + x) - \tan(a - x)}{\tan^{-1}(a + x) - \tan^{-1}(a - x)}$; when $x = 0$, $u = (a^2 + 1)\sec^2 a$.

34. If $u = \left(\sec \dfrac{\pi x}{2}\right)^2 \operatorname{vers} 2\pi x$; when $x = 1$, $u = 8$. $= 4$

35. If $u = x \tan x - \dfrac{\pi}{2} \sin x$; when $x = \dfrac{\pi}{2}$, $u = -1$.

36. If $u = \dfrac{\cos^{-1}(1 - x)}{\sqrt{2x - x^2}}$; when $x = 0$, $u = 1$.

37. If $u = \frac{1 - 3x^2}{2x^2(1 - x^2)} - \frac{\pi}{2x \tan \pi x}$; when $x = 1$, $u = -\frac{3}{4}$.

38. If $u = \frac{\log \tan x}{\log \tan 2x}$; when $x = 0$, $u = 1$.

39. If $u = 2^x \tan \frac{a}{2^x}$; when $x = \infty$, $u = a$.

40. If $u = \frac{a^x \sin ax - b^x \sin bx}{c^x \sin cx - e^x \sin ex}$; when $x = 0$, $u = \frac{a - b}{c - e}$.

41. If $u = \frac{1}{\log(1 + x)} - \frac{1}{x}$; when $x = 0$, $u = \frac{1}{2}$.

42. If $u = \frac{x - 1}{2x^2} + \frac{1}{x(e^{2x} - 1)}$; when $x = 0$, $u = \frac{1}{6}$.

43. If $u = \frac{(x^2 - a^2)^{\frac{3}{2}} + x - a}{(1 + x - a)^3 - 1}$; when $x = a$, $u = \frac{1}{3}$.

44. If $u = \frac{1 - x + \log x}{1 - \sqrt{2x - x^2}}$; when $x = 1$, $u = \pm 1$.

45. If $u = \frac{a - \sqrt{2ax - x^2}}{a - \sqrt[3]{2a^2x - ax^2}}$; when $x = a$, $u = \frac{3}{2}$.

46. If $u = \frac{\sqrt{2a^2 + 2x^2} - 2\sqrt[3]{a^2x}}{x - a}$; when $x = a$, $u = \frac{1}{3}$.

47. If $u = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$; when $x = a$, $u = \frac{16a}{9}$.

48. If $u = \frac{a - x - a \log a + a \log x}{a - \sqrt{2ax - x^2}}$; when $x = a$, $u = -1$.

49. If $u = \frac{(x - c)\sqrt{x - b} + \sqrt{x - c}}{\sqrt{2c} - \sqrt{x + c} + \sqrt{x - c}}$; when $x = c$, $u = 1$.

50. If $u = \dfrac{\sqrt{a^2 + ax + x^2} - \sqrt{a^2 - ax + x^2}}{\sqrt{a + x} - \sqrt{a - x}}$; when $x = 0$, $u = \sqrt{a}$.

51. If $u = \dfrac{a\sqrt[3]{4a^3 + 4x^3} - ax - a^2}{\sqrt{2a^2 + 2x^2} - a - x}$; when $x = a$, $u = 2a$.

52. If $u = \dfrac{x - a + \sqrt{2ax - 2a^2}}{\sqrt{x^2 - a^2}}$; when $x = a$, $u = 1$.

53. If $u = \dfrac{\sqrt{x^2 - a^2} - \sqrt{2ax - x^2} + a}{\sqrt[3]{x^3 - a^3} - \sqrt{x^2 - a^2}}$; when $x = a$, $u = 0$.

54. If $u = \dfrac{x\sqrt{3a^3x - 2x^4} - ax\sqrt[5]{a^4x}}{a - \sqrt[3]{ax^2}}$; when $x = a$, $u = \dfrac{81}{20}a^2$, by the methods of Articles (103) and (104).

55. If $u = \dfrac{\sqrt{x^2 + a} - \sqrt{a^2 + x} + \sqrt{a^2 - x^2}}{\sqrt{a^3 - x^3}}$; when $x = a$, $u = \sqrt{\dfrac{2}{3a}}$, by the methods of Articles (103) and (104).

56. If $u = \dfrac{\sqrt[3]{a^2x + 7a^3} - \sqrt{2ax + 2a^2}}{\sqrt[4]{a^2x^6 + 15a^5x^3} - \sqrt[5]{15a^4x^6 + 17ax^9}}$; when $x = a$, $u = \dfrac{160}{693a}$, by both the methods.

57. In $y^4 - a^2y^2 + 2a^2x^2 - x^4 = 0$; when $x = 0$ and $y = 0$, the values of $\dfrac{dy}{dx}$ are $\pm\sqrt{2}$.

58. In $(y^2 + ax)^2 = x^2(a^2 + 2ax - x^2)$; when $x = 0$ and $y = 0$, the values of $\dfrac{dy}{dx}$ are ± 1 and ∞.

59. In $(y^2 - x^2)^2 = x^3 - 2axy^2$; when $x = 0$ and $y = 0$, the values of $\dfrac{dy}{dx}$ are $\pm\dfrac{1}{\sqrt{2a}}$ and ∞.

CHAP. VII.

MISCELLANEOUS EXAMPLES.

1. If $u=x^2-2ax+b^2$; when $x=a$, u will be a *minimum*, whose value is b^2-a^2.

2. If $u=x^2+ax+b^2$; when $x=-\frac{1}{2}a$, $u=b^2-\frac{1}{4}a^2$, will be a *minimum*.

3. If $u=x^3-9x^2+24x+16$; when $x=2$, $u=36$, a *maximum;* when $x=4$, $u=32$, a *minimum.*

4. If $u=x^3-18x^2+96x-20$; when $x=4$, $u=140$, a *maximum*; when $x=8$, $u=108$, a *minimum.*

5. If $u=3x^4-28ax^3+84a^2x^2-96a^3x+48a^4$; when $x=a$, $u=11a^4$, a *minimum;* when $x=2a$, $u=16a^4$, a *maximum;* and when $x=4a$, $u=-16a^4$, a *minimum.*

6. If $u=12x^5-45x^4+40x^3$; when $x=1$, $u=7$, a *maximum;* when $x=2$, $u=-16$, a *minimum;* and when $x=0$, $u=0$ is neither a *maximum* nor a *minimum.*

7. If $u=x^6-6ax^5+9a^2x^4+a^6$; when $x=0$, $u=a^6$, a *minimum;* when $x=2a$, $u=17a^6$, a *maximum;* and when $x=3a$, $u=a^6$, a *minimum.*

8. If $u=10x^6-12x^5+15x^4-20x^3+20$; when $x=0$, $u=20$, which is neither a *maximum* nor a *minimum;* and when $x=1$, $u=13$, a *minimum.*

9. If $u=\sqrt{a^2+2bx+mx^2}-nx$; when

$$x=-\frac{b}{m}+\frac{n}{m}\sqrt{\frac{ma^2-b^2}{m-n^2}},\quad u=\frac{nb+\sqrt{(ma^2-b^2)(m-n^2)}}{m},$$

a *maximum* or a *minimum*, according as m is less or greater than n^2, provided $m-n^2$ and ma^2-b^2 have the same sign; and when

$$x=-\frac{b}{m}-\frac{n}{m}\sqrt{\frac{ma^2-b^2}{m-n^2}},\quad u=\frac{nb-\sqrt{(ma^2-b^2)(m-n^2)}}{m}$$

a *minimum* or a *maximum* on the same hypothesis: also, when $m-n^2$ and ma^2-b^2 have different signs, the values of x and u become impossible, and consequently by (115) there can be neither a *maximum* nor a *minimum*.

If $m=2$, $n=1$ and $b=0$; when $x=\dfrac{a}{\sqrt{2}}$, $u=\dfrac{a}{\sqrt{2}}$, a *minimum;* and when $x=-\dfrac{a}{\sqrt{2}}$, $u=\dfrac{3a}{\sqrt{2}}$, a *maximum.*

10. If $u=24x^5-135x^4+280x^3-270x^2+120x+180$; when $x=\frac{1}{2}$, $u=199\frac{13}{16}$, a *maximum;* when $x=2$, $u=188$, a *minimum:* and when $x=1$, $u=199$, is neither.

11. If $u=(x-a)^n$; when $x=a$, $u=0$, which is a *minimum* if n be even; but neither a *maximum* nor a *minimum* if n be odd, as appears from (112) and (113).

12. If $u=x^2(a-x)^3$; when $x=0$, $u=0$, a *minimum;* when $x=a$, $u=0$, which is neither a *maximum* nor a *minimum;* and when $x=\dfrac{2a}{5}$, $u=\dfrac{108a^5}{3125}$, a *maximum.*

13. If $u=(x-1)^4(x+2)^3$; when $x=-2$, $u=0$, which is neither a *maximum* nor a *minimum;* when $x=-\dfrac{5}{7}$, $u=\dfrac{12^4 \cdot 9^3}{7^7}$, a *maximum;* and when $x=1$, $u=0$, a *minimum.*

14. If $u=x^m(a-x)^n$; when $x=0$, $u=0$, which is a *minimum* when m is even, but neither a *maximum* nor a *minimum* when m is odd; when $x=a$, $u=0$, which is a *minimum* when n is even, but neither a *maximum* nor a *minimum* when n is odd; when $x=\dfrac{ma}{m+n}$, $u=\dfrac{m^m n^n a^{m+n}}{(m+n)^{m+n}}$, which is a *maximum:* as appears from (114).

15. If $u = \dfrac{a^2 + ax + x^2}{a + x}$; when $x = 0$, $u = a$, a *minimum;* and when $x = -2a$, $u = -3a$, a *maximum.*

16. If $u = \dfrac{(1+x)^3}{(1+x^2)^2}$; when $x = -2 + \sqrt{7}$, $u = \dfrac{17 + 7\sqrt{7}}{16} = 2.220$, a *maximum;* when $x = -2 - \sqrt{7}$, $u = \dfrac{17 - 7\sqrt{7}}{16} = -.0950$, a *minimum,* and when $x = -1$, $u = 0$, is neither.

17. If $u = \dfrac{(x+2)^3}{(x+3)^2}$; when $x = -5$, $u = -\dfrac{27}{4}$, a *maximum;* when $x = -3$, $u = \infty$, a *maximum;* and when $x = -2$, $u = 0$, which is neither a *maximum* nor a *minimum.*

18. If $u = \dfrac{(x-1)^m}{(x+1)^n}$; when $x = \dfrac{n+m}{n-m}$, and m and n are both positive whole numbers of which n is the greater, $u = \dfrac{m^m}{n^n}\left(\dfrac{n-m}{2}\right)^{n-m}$, a *maximum;* but if n be less than m, $u = \dfrac{m^m}{n^n}\left(\dfrac{2}{n-m}\right)^{m-n}$, which will be a *maximum* or a *minimum,* according as $m - n$ is an odd or an even number.

If $m = 3$, and $n = 2$; when $x = -5$, $u = -13\tfrac{1}{2}$, a *maximum.*

19. If $u = \dfrac{x^2 - x + 1}{x^2 + x - 1}$; when $x = 0$, $u = -1$, a *maximum;* and when $x = 2$, $u = \frac{3}{5}$, a *minimum:* for these see (123).

20. If $u = \dfrac{x^2 - 3x + 2}{x^2 + 3x + 2}$; when $x = \sqrt{2}$, $u = 12\sqrt{2} - 17$, a *minimum;* and when $x = -\sqrt{2}$, $u = -12\sqrt{2} - 17$, a *maximum:* as may be proved by (123).

21. If $u = \dfrac{x(x^2 + 1)}{(x^2 - 1)^2 + x^2}$; when $x = 1$, $u = 2$, a *maximum;* and when $x = -1$, $u = -2$, a *minimum:* see Article (123).

22. If $u = \dfrac{x^3 - x}{x^4 - x^2 + 1}$; when $x = \pm \left(\dfrac{1 + \sqrt{5}}{2}\right)$, $u = \pm \frac{1}{2}$, a *maximum* and a *minimum;* and when $x = \pm \left(\dfrac{1 - \sqrt{5}}{2}\right)$, $u = \pm \frac{1}{2}$, a *maximum* and a *minimum:* see Article (123).

23. If $u = \dfrac{(a+x)\,(b+x)}{(a-x)\,(b-x)} = (a+x)\,(b+x)\,(a-x)^{-1}\,(b-x)^{-1}$;

when $x = \sqrt{ab}$, $u = -\left(\dfrac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}}\right)^2$, a *minimum;*

when $x = -\sqrt{ab}$, $u = -\left(\dfrac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}\right)^2$, a *maximum.*

24. If $u = \left(\dfrac{a}{x}\right)^x$; when $x = \dfrac{a}{e}$, $u = e^{\frac{a}{e}}$, a *maximum.*

25. If $u = \sqrt{a^2 + x^2} + \sqrt{a^2 + \left(\dfrac{ab}{x}\right)^2}$;

when $x = \sqrt{ab}$, $u = 2\sqrt{a\,(a+b)}$, a *minimum;*
and when $x = -\sqrt{ab}$, $u = 2\sqrt{a\,(a+b)}$, a *maximum.*

26. If $u = \dfrac{x}{\log x}$; when $x = e$, $u = e$, a *minimum.*

27. If $u = \sin x \cos 2x$; when $x = \sin^{-1}\left(\dfrac{1}{\sqrt{6}}\right)$, $u = \dfrac{2}{3\sqrt{6}}$, a *maximum;* and when $x = \sin^{-1}\left(\dfrac{-1}{\sqrt{6}}\right)$, $u = -\dfrac{2}{3\sqrt{6}}$, a *minimum.*

28. If $u = \cos mx \cos^m (a + x)$; when $x = \dfrac{\pi - a}{m + 1}$,

$u = \cos \dfrac{m}{m+1}(\pi - a) \cos^m \left(\dfrac{\pi + ma}{m+1}\right)$, a *maximum.*

29. If $u = a \sec x + b \operatorname{cosec} x$; when $\sin x = \pm \dfrac{b^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}$,

$u = \pm \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{2}{3}}$, a *minimum* and a *maximum.*

30. If $u = \sin x \operatorname{vers} x$; when $x = \frac{2}{3}\pi$, $u = \frac{3}{4}\sqrt{3}$, a *maximum;* and when $x = 0$, $u = 0$, which is neither a *maximum* nor a *minimum.*

31. If $u = \sin x \cos^2 x$; when $x = 90^0$, $u = 0$, a *minimum;* and when $x = \cos^{-1}\sqrt{\frac{2}{3}}$, $u = \frac{2}{9}\sqrt{3}$, a *maximum.*

32. If $u = \cos x \sin (a - x)$; when $x = \frac{1}{2}a + 45^0$,
$u = -\frac{1}{2}(1 - \sin a)$, a *minimum;* and when $x = \frac{1}{2}a - 45^0$,
$u = \frac{1}{2}(1 + \sin a)$, a *maximum.*

33. If $u = e^x \cos (x - a)$; when $x = a + \frac{1}{4}\pi$, $u = \dfrac{e^{a + \frac{\pi}{4}}}{\sqrt{2}}$, a *maximum;* when $x = a + \frac{5}{4}\pi$, $u = -\dfrac{e^{a + \frac{5\pi}{4}}}{\sqrt{2}}$, a *minimum.*

✓ 34. If $u = \dfrac{e^x}{\sin (x - a)}$; when $x = a + \dfrac{\pi}{4}$, $u = \sqrt{2}e^{a + \frac{\pi}{4}}$, a *minimum;* and when $x = a + \dfrac{5\pi}{4}$, $u = -\sqrt{2}e^{a + \frac{5\pi}{4}}$, a *maximum.*

35. If $u^2 - x^2u + x - x^3 = 0$; when $x = -1$, $u = 1$, a *maximum;* and when $x = -1$, $u = 0$, which is neither a *maximum* nor a *minimum.*

36. If $u^3 - a^2x + x^3 = 0$; when $x = -\dfrac{a}{\sqrt{3}}$, $u = -\dfrac{2^{\frac{1}{3}}}{3^{\frac{1}{2}}}a$, a *minimum;* when $x = \dfrac{a}{\sqrt{3}}$, $u = \dfrac{2^{\frac{1}{3}}}{3^{\frac{1}{2}}}a$, a *maximum.*

37. If $u^2 + 2x^2u + 4x - 3 = 0$; when $x = -\frac{1}{2}$, $u = 2$, a *maximum;* and when $x = 1$, $u = -1$, which is neither a *maximum* nor a *minimum.*

38. If $u^4 - 4xu + x^4 + 2 = 0$; when $x = 1$, $u = 1$, which cannot be considered either a *maximum* or a *minimum,* because the values both *immediately* preceding and following it are impossible; also when $x = -1$, $u = -1$, to which the same observation applies.

39. If $u=\sqrt{a^2+x^2}-x$; when $x=\infty$, $u=0$, which, according to the criterion, is neither a *maximum* nor a *minimum;* but since it cannot become negative, $u=0$, may be considered a *minimum*.

40. If $u^2+mxu+a^2+bx+nx^2=0$;

$$\text{when } x=\frac{2nb+m\sqrt{nb^2+(m^2-4n)\,na^2}}{n\,(m^2-4n)},$$

$$u=\frac{-mb-2\sqrt{nb^2+(m^2-4n)\,na^2}}{m^2-4n}, \text{ a } \textit{maximum;}$$

$$\text{and when } x=\frac{2nb-m\sqrt{nb^2+(m^2-4n)\,na^2}}{n\,(m^2-4n)},$$

$$u=\frac{-mb+2\sqrt{nb^2+(m^2-4n)\,na^2}}{m^2-4n}, \text{ a } \textit{minimum.}$$

If $m=0$, $x=-\frac{b}{2n}$, and $u=\pm\frac{1}{2n}\sqrt{nb^2-4n^2a^2}$, which are respectively a *maximum* and a *minimum*.

If $n=0$, $x=\infty$, and $u=-\frac{b}{m}$, which is neither a *maximum* nor a *minimum*.

If $m^2=4n$, $x=-\frac{m^2a^2+b^2}{m^2b}$, and $u=\frac{m^2a^2-b^2}{mb}$, which is a *minimum*.

41. If $u=2x-x^2\pm(1-x)^{\frac{5}{2}}$; when $x=1$, $u=1$, a *maximum:* when $x=\frac{9}{25}$, $u=\frac{2869}{3125}$, a *minimum* of the first kind; and when $x=1$, $u=1$, which is also a *maximum* of the second kind.

42. If $u=\cos x+\cos 2x+\cos 3x$; when $x=0$, $u=3$, a *maximum;* when $x=\cos^{-1}\left(\frac{-1+\sqrt{7}}{6}\right)$, $u=-\frac{17+7\sqrt{7}}{27}$, a *minimum;* and when $x=\cos^{-1}\left(\frac{-1-\sqrt{7}}{6}\right)$, $u=-\frac{17-7\sqrt{7}}{27}$, a *maximum*.

43. If $u^4-4a^2ux+x^4=0$; when $x=a\sqrt[8]{3}$, $u=a\sqrt[8]{27}$, a *maximum;* and when $x=-a\sqrt[8]{3}$, $u=-a\sqrt[8]{27}$, a *minimum.*

44. If $2aux-(a-b)x^2=a^3$; when $x=\pm\dfrac{a^{\frac{3}{2}}}{\sqrt{a-b}}$, $u=\pm\sqrt{a^2-ab}$, which are both *minima.*

45. If $u=\dfrac{\sqrt{1+x+2a\sqrt{x}}+\sqrt{1+x-2a\sqrt{x}}}{\sqrt{1+x+2a\sqrt{x}}-\sqrt{1+x-2a\sqrt{x}}}$; when $x=1$, $u=\dfrac{1+\sqrt{1-a^2}}{a}$, which is a *minimum.*

46. If $u=a^{x+1}-a^x-x$; when $x=-\dfrac{\log\{(a-1)\log a\}}{\log a}$, $u=\dfrac{1+\log\{(a-1)\log a\}}{\log a}$, a *minimum* or a *maximum* according as a is greater or less than 1.

47. If $u=a(\cos x)^2+b\{\cos(\alpha-x)\}^2$; when

$$x=\frac{\alpha}{2}-\tfrac{1}{2}\tan^{-1}\left(\frac{a-b}{a+b}\tan\alpha\right),$$

$$u=a\left\{\cos\left[\frac{\alpha}{2}-\tfrac{1}{2}\tan^{-1}\left(\frac{a-b}{a+b}\tan\alpha\right)\right]\right\}^2$$

$$+b\left\{\cos\left[\frac{\alpha}{2}+\tfrac{1}{2}\tan^{-1}\left(\frac{a-b}{a+b}\tan\alpha\right)\right]\right\}^2, \text{ a } \textit{maximum}.$$

48. If $u=(\tan x)^m\{\tan(\alpha-x)\}^n$; when

$$x=\frac{\alpha}{2}+\tfrac{1}{2}\tan^{-1}\left(\frac{m-n}{m+n}\tan\alpha\right),$$

$$u=\left\{\tan\left[\frac{\alpha}{2}+\tfrac{1}{2}\tan^{-1}\left(\frac{m-n}{m+n}\tan\alpha\right)\right]\right\}^m\times$$

$$\left\{\tan\left[\frac{\alpha}{2}-\tfrac{1}{2}\tan^{-1}\left(\frac{m-n}{m+n}\tan\alpha\right)\right]\right\}^n, \text{ a } \textit{maximum}.$$

CHAP. VIII.

MISCELLANEOUS EXAMPLES.

It is required by means of Articles (126)—(189) to effect the solutions of the following Theorems and Problems.

1. If x and y be the co-ordinates of a parabola, whose *latus rectum* is a, corresponding to the arc s, then will

$$\frac{ds}{dx} = \tfrac{1}{2}\sqrt{\frac{a+4x}{x}} = \frac{1}{2y}\sqrt{a^2+4y^2};$$

$$\text{and } \frac{ds}{dy} = \sqrt{\frac{a+4x}{a}} = \frac{1}{a}\sqrt{a^2+4y^2}.$$

2. In a common cycloid, the radius of whose generating circle is a, it is required to prove that

$$\frac{ds}{dx} = \sqrt{\frac{2a}{x}} \text{ and } \frac{ds}{dy} = \sqrt{\frac{2a}{2a-x}}.$$

3. If $c^2y = x^3$, be the equation to a curve; then will the secant, passing through two points whose abscissæ are a and b, intersect the axis of x in an angle whose trigonometrical tangent is $\dfrac{a^2+ab+b^2}{c^2}$: required a proof.

4. Find the angles in which a secant drawn through two given points of the curve whose equation is $y = ax^2 - bx + c$, intersects the co-ordinate axes.

5. Determine the angles in which a secant to any curve of the second order intersects the co-ordinate axes.

6. Find the angle contained between the two straight lines drawn from the vertex of an ellipse, to the extremities of the *latus rectum* and axis minor.

7. Find the angle which a straight line, equally inclined to each of the co-ordinate axes, drawn through the focus of an ellipse or hyperbola makes with the distance from the centre.

8. If $p = \frac{dy}{dx}$; then may the inclinations of the rectilineal tangent to the axes of co-ordinates be determined from the equations $X = \sin^{-1}\frac{p}{\sqrt{1+p^2}} = \cos^{-1}\frac{1}{\sqrt{1+p^2}}$.

9. What are the inclinations to the axes, of the rectilineal tangents at the extremities of the *latus rectum* of each of the conic sections referred to rectangular co-ordinates?

10. At what angle does the rectilineal tangent of the curve whose equation is $ay^2 = a^2x + x^3$, meet the axis of x, at a point where $x = a$?

11. Find the angles which a tangent to the cycloid makes with the axes; and prove it to be parallel to the corresponding chord of the generating circle.

12. Determine the inclinations to the axes, of a tangent to the trochoid whose equation is $y = ma \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2}$, at a point whose abscissa is $\frac{(m+1)a}{m}$.

13. If any ordinate of the curve whose equation is $y^3 - 3axy + x^3 = 0$ be drawn; prove that the sum of the trigonometrical tangents of the angles which the lines touching the curve at the points of intersection make with the axis of $x = 0$.

14. Find the angles at which the cissoid of *Diocles* is inclined to the axes, where it intersects the generating circle.

15. Determine the points in a circle at which it is inclined to the axis of x at the angles 30^0 and 60^0; and find the distance between them.

16. Required the inclination to the axis of x, of the curve whose equation is $\log(a + \sqrt{a^2 - y^2}) = \frac{x + \sqrt{a^2 - y^2}}{a} + \log y$, at a point where $y = \frac{a}{\sqrt{2}}$.

17. In the curve whose equation is $a^3y = x(b^3 - x^3)$, it is required to find the points at which the rectilineal tangents make with the axis of x, the angles of $30^0, 45^0, 60^0$ and 75^0.

18. If $\frac{x^2}{ay} + \frac{y^2}{bx} = 1$, be the equation to a curve; it is required to draw a tangent to it, and to find the points at which it is parallel to the axes.

19. In the curve whose equation is $ay = x\sqrt{a^2 - x^2}$, find the points at which the tangents are equally inclined to both the axes.

20. Shew that the rectilineal tangent to the curve whose equation is $(x-2)y = (x-1)(x-3)$, can never be equally inclined to both the axes.

21. Prove that the curve whose equation is $(a^2+x^2)y = b^2x$, intersects the axis of x at the origin of co-ordinates in an angle whose trigonometrical tangent is $\frac{b^2}{a^2}$.

22. Required the angles in which the curve whose equation is $y^2 + ay = ax - x^2$, intersects the co-ordinate axes.

23. Find all the angles of intersection with the axes, of the curve whose equation is $y = a\sin x + b\sin 2x$.

24. Find all the angles in which the curve whose equation is $(x^2+y^2)^2 = 2a^2(x^2 - y^2)$, cuts the co-ordinate axes.

25. Prove that the curve whose equation is $4by = 4abx - (a^2+1)x^2$, cuts the axis of x in two angles supplemental to each other.

26. The equation to a curve being $(a-x)^2(x^2+y^2) = a^2y^2$; it is required to find the angles in which it intersects the axes of x and y.

27. Two given rectangular hyperbolas have a common vertex and their axes perpendicular to each other: find the angles in which they intersect each other.

28. A tangent is drawn to a given ellipse which cuts the circle described on its axis major at an angle of 45°: determine the co-ordinates of the points in which it intersects the circle, and thence shew that no such line can be drawn if the eccentricity be less than $\frac{1}{\sqrt{2}}$.

29. The centre of an ellipse coincides with the vertex of a common parabola, and the axis major of the ellipse is perpendicular to the axis of the parabola; find the eccentricity of the ellipse when it cuts the parabola at right angles.

30. A parabola and hyperbola have the same vertex and principal axis; it is required to draw a tangent to the former which shall cut the latter in a given angle.

31. The vertex of a parabola is A and the co-ordinates of any point in it are AM, MP: in MP a point Q is taken always equidistant from A and P; it is required to find the angles in which the locus of Q cuts the axis and the arcs of the parabola.

32. Find the angles in which two concentric ellipses having their axes in the same directions intersect each other; and explain the result when the ellipses become similar.

33. The focus of a given parabola is situated in the centre of a given circle; find their angles of intersection and the length of the portion of the corresponding tangent of the parabola included within the circle.

34. Find the same when any given point in the axis of the parabola coincides with any given point in the diameter of the circle, their principal axes being in the same directions.

35. Prove that the rectilineal tangent to the cissoid of *Diocles* cuts off from the axes of x and y, portions represented by $-\frac{ax}{3a-x}$ and $-a\left(\frac{x}{2a-x}\right)^{\frac{3}{2}}$ respectively.

36. In the curve whose equation is $y^m = ax^{m-1} + x^m$, the portions of the co-ordinate axes cut off by the rectilineal tangent are $\frac{ax}{(m-1)a+mx}$ and $\frac{ay}{m(a+x)}$: required a proof.

37. Find the distances from the origin at which the rectilineal tangent of the curve defined by the equation $ay^{m+n} = bx^m (a+x)^n$, intersects the co-ordinate axes.

38. Determine the points in which the rectilineal tangent to the curve defined by the equation $x^2y^2 = (a^2 - x^2)(b+x)^2$, intersects the axes.

39. In the logarithmic curve whose equation is $y = a^x$, the distances from the origin at which the rectilineal tangent meets the axes of x and y are $\dfrac{1 - x\log a}{\log a}$ and $(1 - x\log a)\,y$, and the subtangents are $\dfrac{1}{\log a}$ and $xy\log a$: required a proof.

40. Find the points of intersection and the angles of inclination, of the rectilineal tangent to the co-ordinate axes, when $x^2 + y^2 = \sqrt{x^2 - y^2}$ is the equation to the curve.

41. Determine where a rectilineal tangent to the quadratrix of *Dinostratus* defined by the equation $y = (1-x)\tan\frac{1}{2}\pi x$, intersects the co-ordinate axes.

42. Find the subtangents in the quadratrix of *Tschirnhausen* whose equation is $y = \sin\frac{1}{2}\pi x$.

43. Determine where the rectilineal tangent to the cycloidal curve defined by the equations $x = a(1 - \cos\theta)$ and $y = a(m\theta + n\sin\theta)$, intersects the co-ordinate axes.

44. Find expressions for the subtangents of the cardioide defined by the equations $x = a(2\cos\theta - \cos 2\theta)$ and

$$y = a(2\sin\theta - \sin 2\theta).$$

45. If the equation of a curve be $xy(x+y) = a^3$; it is required to shew that the rectilineal tangent meets the axes at distances from the origin represented by

$$-\frac{3a^3}{2xy + y^2} \text{ and } \frac{3a^3}{2xy + x^2}.$$

46. If the focus of a parabola be the origin of co-ordinates, it is required to prove that the perpendicular from it upon the tangent $= \frac{1}{2}\sqrt[4]{a^2(x^2+y^2)}$.

47. Find the values of the perpendiculars from the vertices of an ellipse and hyperbola upon the tangent at any point; and determine what they become for a circle and equilateral hyperbola.

48. Required the same from the centres and foci generally, and for the above-mentioned particular cases.

49. Find the parts of the tangent of an ellipse intercepted between the point of contact and the co-ordinate axes through the centre: and thence prove that their rectangle is invariable when the ellipse becomes a circle.

50. Determine the same in an hyperbola generally; and deduce a similar conclusion when the hyperbola becomes equilateral.

51. If a diameter and a tangent at the extremity of the *latus rectum* of a parabola be considered the co-ordinate axes; it is required to find the points in which the tangent to any other point meets them, and also the perpendicular upon it from the origin.

52. In an ellipse referred to a pair of conjugate diameters, one of which bisects the angle between the semi-axes, it is required to find the subtangents and the lengths of the corresponding tangents.

53. Determine the angles in which the rectilineal tangent to the cissoid of *Diocles*, at points whose subtangent is equal to $\frac{1}{2}a$, intersects the co-ordinate axes.

54. In the trochoid whose equation is

$$y = ma \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2},$$

find the subtangents when the tangent is equally inclined to the co-ordinate axes; and determine for what values of m this is possible.

55. The equations to the tangents of an ellipse and hyperbola whose centres are the origins, are $a^2yy' \pm b^2xx' = \pm a^2b^2$; and the portions which they cut off from the axes of x and y are $\frac{a^2}{x}$ and $\frac{b^2}{y}$ respectively: required a proof.

56. Find the equations to the tangents of an ellipse and hyperbola in which the vertices are the origins of co-ordinates; and determine the portions they cut off from the co-ordinate axes.

57. The tangent to the cissoid of *Diocles* is defined by the equation $(2a-x)^{\frac{3}{2}}y' = (3ax' - xx' - ax)\,x^{\frac{1}{2}}$: required a proof.

58. Prove that the equation to the tangent of the conchoid of *Nicomedes* is

$$x^2\sqrt{b^2-x^2}\,y' + (ab^2+x^3)\,x' - 2ab^2x - b^2x^2 + ax^3 = 0\,;$$

and find what this becomes when $x=a=b$.

59. The witch is defined by the equation $xy = a\sqrt{ax-x^2}$; it is required to prove that the equation to its tangent will be

$$2x\sqrt{ax-x^2}\,y' + a\,(ax' + 2x^2 - 3ax) = 0,$$

which cuts the co-ordinate axes at distances from the origin expressed by $\frac{x}{a}(3a-2x)$ and $\frac{a\,(3a-2x)}{2\sqrt{ax-x^2}}$.

60. Find the equation to the tangent at any point of a cycloid: prove that it is parallel to the corresponding chord of the generating circle, and determine what portion of it is intercepted between the co-ordinate axes.

61. Find the equation of the tangent to the companion of the cycloid defined by the equations $x = a$ vers θ and $y = a\theta$.

62. Required the equation of the tangent of the curve defined by $y = (\sqrt{a} - \sqrt{x})^2$; and by means of it prove that the sum of the portions of the co-ordinate axes cut off by the tangent is of an invariable magnitude.

63. Find the equation to the tangent of the curve defined by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; and from it shew that the length of the tangent intercepted between the co-ordinate axes is invariable.

64. Determine the equation to the tangent of the curve defined by $\frac{x+\sqrt{a^2-y^2}}{a} = \log\frac{a+\sqrt{a^2-y^2}}{y}$; and thence prove

that the part of it between the point of contact and the axis of x is of invariable length.

65. If the equation of a curve be $x = \frac{y^2 - a^2}{4a} + \frac{a}{2} \log \frac{a}{y}$; find the equation of its tangent, and thence prove that the difference between the tangent and subtangent belonging to the axis of x is invariable.

66. Of the curve defined by $e^{\frac{y}{a}} = x^2 - a^2$, find the equation to the tangent; and thence prove that the sum of the tangent and subtangent varies as the rectangle of the corresponding co-ordinates.

67. Find the equation of the tangent to the curve defined by $(m-1)^{m-1} y^m = m^m a^{m-1} x$: and thence shew that if AD be drawn from the origin perpendicular to the axis of x and meet the tangent PT in D, $AD^m = a^{m-1} AT$.

68. Required the equation to the tangent of the curve whose equation is $x \sqrt{2} (x^2 + y^2) = a (x^2 - y^2)$; and find where it intersects the axes of co-ordinates.

69. In the curve whose equation is $(1 + \log x) y = x$, find the equation to the tangent; and thence prove that at any point, the subtangent : the distance of the intersection of the tangent with the axis of x from the origin :: the abscissa : the ordinate.

70. From a point in a curve whose co-ordinates are x and y, a straight line is drawn making with the curve an angle whose tangent is a; prove that the equation to the line will be $y' - y = \frac{a + p}{1 - ap} (x' - x)$, where $p = \frac{dy}{dx}$.

71. If straight lines be drawn perpendicular to the major axis of an ellipse at its extremities; find the points in which a tangent at any point of the curve meets them, and prove that the rectangle of the parts cut off is invariable.

72. In the curve defined by $\log \sqrt{x^2 + y^2} = a \tan^{-1} \frac{y}{x}$, find the equation to the tangent; and prove that the angle in which

it intersects the line drawn from the origin to the point of contact is invariable.

73. Draw a tangent to the curve whose equation is $2a^m xy^{m-1} = y^{2m} - a^{2m}$; and prove that the distance of the point of contact, from a given point in the axis of x, varies as the segment of the said axis between that point and the tangent.

74. If any tangent be drawn to an ellipse, and four perpendiculars be drawn to the axis major, from the centre of the ellipse, the two extremities of the axis major and the point of contact, these four perpendiculars are proportionals.

75. Find the general equation to the tangent at any point of a curve when the co-ordinates are oblique; and apply it to an ellipse where the co-ordinate axes are parallel to any pair of conjugate diameters.

76. If a and b be the semi-axes of an ellipse or hyperbola; the product of the tangents of the angles at which any system of conjugate diameters are inclined to the axis major is $-\frac{b^2}{a^2}$ or $\frac{b^2}{a^2}$: required a proof.

77. If a system of conjugate diameters of a conic section be produced to meet a tangent whose position is given; the rectangle of the parts of the tangent intercepted between those diameters and the point of contact, is independent of their position: required a proof.

78. The equation to a straight line drawn from the point α, β of a curve whose equation is $y = f(x)$, perpendicular to its tangent is $y' - \beta = -\frac{dx}{dy}(x' - \alpha)$: required a proof.

79. In the ellipse and hyperbola referred to the centres and principal axes; prove that the equations to the perpendiculars upon the tangents from the points α, β of the curves are

$$(x' - \alpha)\, a^2 y \pm (y' - \beta)\, b^2 x = 0.$$

80. The point of intersection of the rectilineal tangent with the perpendicular upon it from a point α, β of any curve

may be found from the equation $x' = \frac{p^2x + a - p(y - \beta)}{1 + p^2}$: required a proof.

81. If a perpendicular upon the tangent from the focus of an ellipse be drawn; prove that the distance of the point of their intersection from the centre of the curve is invariable.

82. Given the length of the perpendicular from the centre of an ellipse upon its tangent; to find the point of contact.

83. In an hyperbola, given the length of the perpendicular from either focus upon the tangent; to determine the point of contact.

84. In the lemniscata of *Bernoulli* whose equation is $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$, find the equation to the rectilineal tangent which shall make equal angles with the co-ordinate axes.

85. Required the equation to the tangent making equal angles with the axes of a curve whose equation is $x^3 + y^3 = a^3$; and find where it meets them.

86. In the curve defined by $x^2y^2 = (a^2 - x^2)(x - b)^2$, it is required to find the equations to the tangents inclined to the axis of x at angles whose trigonometrical tangents are $\pm\sqrt{\frac{a^2}{b^2} - 1}$: and to determine the points of contact.

87. Required the equations to the tangents inclined at angles of 30^0 to the axes of the ellipse defined by the equation $y^2 - 2xy + 3x^2 - 2y - 4x + 5 = 0$; and find their points of intersection with the axes.

88. Tangents to an ellipse, from a given point α, β without it, may be drawn by means of the equations

$$a^2y^2 + b^2x^2 = a^2b^2 \text{ and } a^2\beta y + b^2\alpha x = a^2b^2;$$

required a proof.

89. It is required to prove that the equation to the straight line joining the points of contact as determined by the expressions in the last example is $a^2\beta y + b^2\alpha x = a^2b^2$.

90. The equations for determining the tangents to an hyperbola, drawn from a given point α, β without it, are

$$a^2y^2 - b^2x^2 = -a^2b^2 \text{ and } a^2\beta y - b^2\alpha x = -a^2b^2;$$

and that of the straight line joining the points of contact is $a^2\beta y - b^2\alpha x = -a^2b^2$: required a proof.

91. From a given point without a circle, two equal straight lines may be drawn to touch it: prove this, and find their inclination to each other.

92. If from any point in the directrix of a parabola, two tangents be drawn to the curve, they shall be at right angles to each other: required a proof.

93. From a given point in the axis of the curve whose equation is $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$, it is required to draw a tangent to it; and to find the limitations under which this is possible.

94. From a point whose co-ordinates are a and $2a$, it is required to draw a tangent to the curve whose equation is $27ay^2 = 4(x - 2a)^3$; and to determine its length.

95. Find the angle included between two tangents to an ellipse drawn from a given point without it; and determine the distance from its centre at which this is a right angle.

96. Find the portions of the axes, through the vertex of an ellipse or hyperbola, cut off by the normal at any point of the curve.

97. Find the points where the normal to a cycloid meets the co-ordinate axes, and its lengths intercepted between them and the point of contact.

98. Determine the portions of the axes, through the centre of an ellipse or hyperbola, cut off by the normal at any point; and prove that the rectangle of the parts of the normal intercepted by them is invariable.

99. Find the equation to the normal of the curve defined by $x^2 + y^2 = a(x - y)$; and determine the position of a fixed point through which it always passes.

100. Required the equation to the normal of the conic section defined by $y^2 = mx + nx^2$; and thence deduce the values of the subnormals on the axes of x and y for the parabola, ellipse and hyperbola.

101. Find the equation to the normal of the curve defined by $\frac{x^2}{ay} + \frac{y^2}{bx} = 1$; and determine the distances from the origin at which it intersects the co-ordinate axes.

102. Determine the equation to the normal of the curve defined by $2y = a\,(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$; and the portions of it intercepted between the curve and the axes of co-ordinates.

103. Prove that the equation to the normal of an hyperbola between the asymptotes is $abxy_{,} - x^3 x_{,} - a^2 b^2 + x^4 = 0$; and thence construct it.

104. Find the equation to the normal of the cissoid of *Diocles;* and thence determine where the normal at the point of its intersection with the generating circle meets the axes.

105. Required the equation to the normal of the curve defined by $y^{\frac{1}{2}} = ax^{-\frac{1}{2}} + x^{\frac{1}{2}}$, which shall make equal angles with the axes of co-ordinates.

106. Determine the equation of the normal of a conic section which shall pass through a given point α, β; and find the points of intersection with the axes.

107. The equation to the straight line drawn from the point α, β of a curve perpendicular to the normal is

$$y_{,} - \beta = -\frac{dy}{dx}(x_{,} - \alpha):$$

required a proof.

108. The intersection of the normal with the perpendicular upon it from any point α, β of the curve may be determined from the equation $x_{,} = \frac{p^2 \alpha + x - p\,(\beta - y)}{1 + p^2}$: required a proof.

109. If a straight line be drawn from the origin of the co-ordinates to any point of a curve, another straight line

equally inclined to the normal will cut off from the axis of x a portion represented by $\dfrac{2(x+py)(px-y)}{2px-(1-p^2)y}$: required a proof.

110. Determine the limits of the length of the normal in each of the conic sections, by means of the general equation.

111. Express the co-ordinates of a point in any curve in terms of the angle which the normal at that point makes with the axis of x.

112. If θ be the angle made by the normal with the major axis of an ellipse; then will the co-ordinates from the centre be expressed by

$$\frac{a^2\cos\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} \text{ and } \frac{b^2\sin\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}:$$

required a proof.

113. On the same supposition, it is required to prove that the subtangent and subnormal are represented by

$$\frac{b^2\sin\theta\tan\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} \text{ and } \frac{b^2\cos\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}.$$

114. The same notation being retained, the tangent and normal will be expressed by

$$\frac{b^2\tan\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} \text{ and } \frac{b^2}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}$$

respectively: required a proof.

115. If the equation of a curve be $a^2y^2=b^2(2ax+x^2)$, and the notation of Article (151) be retained; then will $AW=a$ and $AZ=b$: required a proof.

116. If $y^3=ax^2-x^3$ be the equation of a curve; then will $AW=-\frac{1}{3}a$ and $AZ=\frac{1}{3}a$: required a proof.

117. If $(x-1)y=(x+1)x$ be the equation of a curve; then will $AW=2$ and $AZ=2$: required a proof.

118. If $(x-a)y^2=(x+a)x^2$ be the equation of a curve; then will $AW=a$ and $AZ=\pm a$: required a proof.

119. If $y^m - ax^{m-1} - x^m = 0$ be the equation of a curve; then will $AW = \frac{a}{m}$ and $AZ = \frac{a}{m}$: required a proof.

120. In the curve whose equation is $ay^{m+n} = bx^m (a+x)^n$, it is required to prove that

$$AW = \frac{n}{m+n} a \text{ and } AZ = \frac{n}{m+n} (a^{m+n-1} b)^{\frac{1}{m+n}}.$$

121. If the equation to a curve be $x^3 + y^3 = a^3$; then will $AW = 0$ and $AZ = 0$, and the asymptote passes through the origin, making 135^0 with the axis of x: required a proof.

122. If $(a - x) y^2 = (a + x) x^2$ be the equation of a curve; then will $AW = -a$ and $AZ = \pm \infty$, and the asymptote is parallel to the axis of y: required a proof.

123. If $(a^2 - x^2) y^2 = (a^2 + x^2) x^2$ be the equation of a curve; then will $AW = \pm a$ and $AZ = \pm \infty$, so that the asymptotes are parallel to the axis of y: required a proof.

124. If the equation to a curve be $(a+c+x) y = b (c+x)$; then will $AW = \infty$ and $AZ = b$ when x is infinite, and the asymptote is parallel to the axis of x: also, $AW = a + c$ and $AZ = \infty$ when y is infinite, and the asymptote is parallel to the axis of y: required a proof.

125. If $y^2 = \frac{x + 2a}{x + a}$ be the equation of a curve; then will $AW = \infty$ and $AZ = \pm 1$, or there are two asymptotes parallel to the axis of x and equi-distant from it: again, $AW = a$ and $AZ = \infty$, or there is another asymptote parallel to the axis of y: required a proof.

126. Apply the principles of Article (151) to determine the position of the rectilineal asymptote of the curve whose equation is $x^5 + a^3 xy - y^5 = 0$.

127. Find the equation to the rectilineal asymptote of the hyperbola defined by the equation $\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1$.

128. If the equation of a curve be $xy = a\sqrt{ax - x^2}$; then will $y' = -\infty$ or $x' = 0$, be the equation to its rectilineal asymptote: required a proof.

129. The equation to the rectilineal asymptote of the curve defined by $(x-2)\,y = (x-1)\,(x-3)$, is $y' - x' + 2 = 0$: required a proof.

130. If the equation of a curve be $(x+1)\,y = (x-1)\,x$; then will the equation to its rectilineal asymptote be $y' - x' + 3 = 0$: required a proof.

131. If the equation of a curve be $x^2 - y^2 - a^2 = 0$; it is required to prove that the equations of its rectilineal asymptotes are $y' \mp x' = 0$.

132. The equation of a curve being $y^3 - ax^2 + x^3 = 0$; it is required to shew that the equation to its rectilineal asymptote is $3\,(x' + y') - a = 0$.

133. The equation of a curve is $ay^3 + x^3y - ax^3 = 0$; then the equation to its rectilineal asymptote is $y' + a = 0$: required a proof.

134. If the equation of a curve be $(x^2 - 1)\,y = (x^2 + 1)\,x$; it is required to shew that the equations to its rectilineal asymptotes are $x' = \pm 1$ and $y' = x'$.

135. If the equation of a curve be $y^3 - 2xy^2 + x^2y - a^3 = 0$; then will those of the rectilineal asymptotes be $y' - x' = 0$ and $y' = 0$: required a proof.

136. The equation of a curve is $xy^2 - y = ax^3 + bx^2 + cx + e$; then the equations to its rectilineal asymptotes are

$$y' = x'\sqrt{a} + \frac{b}{2\sqrt{a}},\quad y' = -x'\sqrt{a} - \frac{b}{2\sqrt{a}},\ \text{and}\ x' = 0:$$

required a proof.

137. Find the equations to the rectilineal asymptotes of the curves defined by the following equations:

(1) $4y^2\,(a^2 - x^2) = (2a^2 - x^2)^2$: (2) $x^2y^2 = a^2 - x^2 + (x - b)^2$:

(3) $(2ax - x^2)\,(x^2 + y^2) = b^2x^2$: (4) $(x+1)^2y = x^2\,(x - 1)$:

(5) $(a^2 + x^2)\,y^2 = (ax - x^2)^2$: (6) $y^4 - x^4 - a^2xy = 0$.

138. The hyperbolic asymptote of the curve whose equation is $x^4 - y^4 + a^2xy = 0$, is defined by $y' = x' + \frac{a^2}{4x'}$: required a proof.

139. The hyperbolic asymptotes of the curve whose equation is $y^3 - 2xy^2 + x^2y - a^3 = 0$, are defined by the equations $y' - x' \mp \sqrt{\frac{a^3}{x'}} = 0$: required a proof.

140. In the curve whose equation is

$$xy^2 - a^2y = x^3 + 2x^2 + x + b,$$

the hyperbolic asymptotes are defined by the equations

$$y' = x' + 1 + \frac{1}{2x'} \text{ and } y' = -x' - 1 - \frac{1}{2x'}:$$

required a proof.

141. In the curve whose equation is $y^3 - 3axy + x^3 = 0$, it is required to prove that the rectilineal and hyperbolic asymptotes will be defined by the equations $x' + y' + a = 0$, and $x'y' + x'^2 + ax' + a^2 = 0$, respectively.

142. The curve whose equation is

$$y^2 - 2xy + 3x^2 - 2y - 4x + 5 = 0,$$

is concave or convex towards the axis of x, according as the greater or less value of y is considered: required a proof.

143. In the curve whose equation is $(x - 1)y = (x + 1)x$, it is required to prove that the branches have their concavities upwards or downwards according as the abscissa is greater or less than 1.

144. The curve whose tangent is of an invariable magnitude is always convex towards both the co-ordinate axes: required a proof.

145. The branches of the curve whose equation is $(a + x)y = a^2 + ax + x^2$, are convex towards both the co-ordinate axes: required a proof.

146. Prove that the curve defined by the equation

$$(y-a)^3 = a^3 - 2a^2x + ax^2,$$

is always concave towards the axis of x; and find how it is situated with respect to the axis of y.

147. Required the direction of the curvature of the curve whose equation is $(a^2 - x^2)\, y = a^3$.

148. Find whether the curve whose equation is $(y-b)^2 = x(x-a)^2$, is convex or concave towards the co-ordinate axes.

149. Find the direction of the curvature of each of the curves defined by the following equations:

(1) $a^2y^2 = b^2x(x-2a)$: (2) $(a-x)\, y^2 = (a+x)\, x^2$:
(3) $(x+a)\, y^2 = b^2(x+2a)$: (4) $y^3 + x^3 = 3axy$:
(5) $y^4 + 2axy^2 = ax^3$: (6) $(x^2+y^2)^3 = 4a^2x^2y^2$:
(7) $xy^2 = ab^2 - b(2a+b)\, x + (a+2b)\, x^2 - x^3$:
(8) $y^4 + 2x(x-3a)\, y^2 - 2ax^3 + a^2x^2 = 0$:
(9) $x^4 - ax^2y + \frac{1}{5}a^2y^2 - ay^3 = 0$.

150. Determine whether the curve whose equation is $(a-x)^2(x^2+y^2) = a^2y^2$, is convex or concave towards the axes, at the points where the values of x are $\frac{1}{2}a$ and $\frac{3}{2}a$.

151. If $y \propto x^m$ in one curve and $y \propto x^n$ in another, m being greater than n; it is required to prove that the former curve always falls between the latter and its rectilineal tangent.

152. Determine the conical parabola which shall have contact of the second order with an ellipse at a given point, and have its axis parallel to the axis major of the ellipse.

153. Define the circle of curvature of the curve whose equation is $2y = e^x + e^{-x}$; and prove that its radius is equal to the normal.

154. Determine the position and magnitude of the circle of curvature, when the equation of the curve is $x^2 - ax = ay - y^2$.

155. If x', y' be the co-ordinates of the circle of curvature at a point x, y of any curve: it is required to prove that the equation to the circle is

$$\{(x'-x)q+(1+p^2)p\}^2+\{(y'-y)q-(1+p^2)\}^2=(1+p^2)^3.$$

156. Required the equation of the circle of curvature of the curve defined by the equation $\dfrac{y}{a}=\cos^{-1}\left(\dfrac{a-x}{a}\right)$.

157. Determine the equation of the circle of curvature of the curve defined by $x=\sec 2y$; and thence find its magnitude and the position of its centre.

158. Find the equation to the circle of curvature at any point of the tractrix.

159. If $y^{\frac{1}{m}}\dfrac{dy}{dx}=\sqrt{a^{\frac{2}{m}}-y^{\frac{2}{m}}}$ be the differential equation of a curve, it is required to shew that the radius of curvature at any point is m times as great as the normal.

160. Determine the radius and chords of curvature of the curve, whose ordinate is equal to the circular arc of which its abscissa is the versed sine.

161. Required the radius and chords of curvature parallel to the axes, of the curve traced out by taking a portion of the radius of a circular arc drawn through its extremity, always equal to the sine of the arc.

162. In any curve, find an expression for the distance of the centre of curvature from any given point a, b; and also from the vertex.

163. Determine expressions for the portions of the axes of any curve included within the circle of curvature; and apply them to the case of the common parabola.

164. Apply the expression for the chord of curvature through a given point, to find the diameter and principal chords of curvature in each of the conic sections.

165. If θ be the angle which the normal at any point of a curve makes with the axis of x, then will the radius of curvature at that point be expressed by $\frac{dy}{d\sin\theta}$ or $\frac{dx}{d\cos\theta}$: required a proof.

166. Apply the expressions above written to prove that in an ellipse the radius of curvature is equal to

$$\frac{a^2b^2}{(a^2\cos^2\theta + b^2\sin^2\theta)^{\frac{3}{2}}}.$$

167. Determine the magnitude of the radius and the co-ordinates of the centre, of the circle of curvature in each of the curves defined by the following equations:

(1) $y^3 = a^2x$: (2) $(1+x^2)\,y = x$:
(3) $a^3y = x\,(b^3 - x^3)$: (4) $a^3y = x^2\,(x+b)^2$:
(5) $a\,(y-b)^2 = x\,(x-a)^2$: (6) $ay^2 = x\,(x+b)^2$:
(7) $(a-x)\,y^2 = (a+x)\,x^2$: (8) $a^2y^2 = x^2\,(a^2 - x^2)$:
(9) $x^3 + y^3 = a^3$: (10) $y = \sin x$: (11) $y = \tan x$.

168. Determine the radius of curvature of the curve whose equation is $y^3 = ax^2 - x^3$, at the points where $x = 0$ and $x = a$.

169. Find that point in a parabola at which the curvature is one fourth part of the greatest curvature.

170. If a circle of curvature to the vertex of a parabola be described, and another circle touch that and the arcs of the parabola, and so on continually: it is required to prove that the radii of these circles are as the odd numbers 1, 3, 5, 7, &c.

171. Find the points in which the circles of curvature at the extremities of the principal axes of an ellipse intersect each other, and also the angles at which the circles are there inclined to each other. Is the intersection always possible?

172. Find the evolute of the curve defined by the equation $y^2 - ax - \frac{1}{2}a^2 = 0$; and determine whether it meets the involute.

173. The evolute of the curve whose equation is

$$a^2y^2 - b^2x^2 + a^2b^2 = 0,$$

is defined by the equation $(a\alpha)^{\frac{2}{3}} - (b\beta)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$: required a proof.

174. The equation to the evolute of the catenary is

$$\beta = a \log \left(\frac{\alpha + \sqrt{\alpha^2 - 4a^2}}{2a}\right) - \frac{\alpha\sqrt{\alpha^2 - 4a^2}}{2a},$$

if the distance of the vertex of the catenary from the origin of co-ordinates be a: required a proof.

175. The evolute of the tractrix is defined by the equation $\alpha = a \log \left(\frac{\beta + \sqrt{\beta^2 - a^2}}{a}\right)$; required a proof.

176. Find the equation to the tangent of the curve defined by $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$; and prove it to be a normal to the curve defined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

177. Required the equation to the tangent of a cycloid, and prove this line to be perpendicular to the tangent of its involute at the point of concourse.

178. Required the length of any portion of the arc of the evolute of the common parabola.

179. Determine the length of any portion of the arc of a common cycloid, the radius of whose generating circle is given.

180. In the curve whose equation is $ay = x\sqrt{x^2 - a^2}$, it is required to prove that the square of the distance of any point in the curve from the origin is equal to half the rectangle of the corresponding abscissa, and the line intercepted between the vertex and the normal.

181. In the curve whose equation is $m(e^{-\frac{y}{m}}-1)=x-y$, it is required to prove that the portion of the ordinate included between the curve and a straight line through the origin making equal angles with the axes varies as the cotangent of the curve's inclination to the axis of x.

182. Describe a circle with a given centre so as to touch a given parabola, both when the point is within and without the parabola.

183. A straight line drawn through the focus of any conic section to the point in which a diameter meets the directrix will be perpendicular to a tangent at either extremity of that diameter: required a proof.

184. If from any point in a line parallel to the axis of a common parabola, two tangents be drawn to the curve; it is required to prove that the sum of the cotangents of their inclinations to the axis of x is invariable.

185. If $y=mx+c\sin\frac{\pi x}{a}$ be the equation to a curve; it is required to shew that the sum of the tangents of the angles in which two ordinates at the distance a from each other cut the curve, is constant.

186. If with a radius equal to the line joining the extremities of the axes of an ellipse, a concentric circle be described, two tangents drawn to the ellipse from any point in the circumference of this circle will be at right angles to each other: required a proof.

187. If with the co-ordinates of any point in an elliptic quadrant as semi-axes a concentric one be constructed, this will be touched by the chord of the first: required a proof.

188. Shew that the ellipse whose equation is

$$y^2-4ax+2x^2=0,$$

always cuts at right angles the parabola whose equation is $y^2=m(a-x)$, whatever be its *latus rectum*.

189. Prove that the part of the tangent of an hyperbola intercepted between the asymptotes is equal to the diameter to which it is parallel, and is bisected by the point of contact.

190. If two equal parabolas have a common axis, prove that a straight line touching the interior and bounded by the exterior will be bisected in the point of contact.

191. Two parabolas having a common axis, it is required to find a point in one of them, from which two tangents drawn to the other shall include a given angle.

192. A curve is constructed by cutting off from the ordinate of a circle a portion equal to the difference between its abscissa and ordinate: find the angles in which it intersects the diameter and the circumference of the circle.

193. If a straight line equal to the sum of the semi-axes of an ellipse have its extremities in the axes and cut the curve, and the parallelogram of which this is the diagonal be completed; it is required to prove that the line drawn from the angle of the parallelogram to the point of intersection will be a normal to the ellipse.

194. If between a rectangular hyperbola and its asymptotes any number of concentric elliptic quadrants be inscribed, the rectangle of their axes will be invariable: required a proof.

195. The ordinate MP of an ellipse whose major axis is AB is bisected in Q and AQ is joined and produced so as to meet the tangent at B in T: prove that the straight line TP will be a tangent to the curve.

196. An ellipse and hyperbola being constructed on the same axes, if from any point in one of the curves two tangents be drawn to the other, the straight line which passes through the points of contact will be a tangent to the first curve: required a proof.

197. AP is a portion of a common parabola, PT a tangent at P, PG a normal and TR a perpendicular to the axis at T: prove that if GP be produced to meet TR in R, GR will be equal to the radius of curvature at P.

CHAP. IX.

MISCELLANEOUS EXAMPLES.

By means of the principles laid down in Articles (190)—(212), it is required to prove and solve the following Theorems and Problems.

1. If θ and r be the co-ordinates of the logarithmic spiral corresponding to the arc s, it is required to shew that

$$\frac{ds}{d\theta} = a^\theta \sqrt{1 + (\log a)^2} = r\sqrt{1 + (\log a)^2}\,; \quad \frac{d}{dr} = \frac{\sqrt{1 + (\log a)^2}}{\log a}.$$

2. In the lemniscata of *Bernoulli* whose equation is $r^2 = a^2 \cos 2\theta$, it is required to prove that

$$\frac{ds}{d\theta} = \frac{a}{\sqrt{\cos 2\theta}} = \frac{a^2}{r}\,; \text{ and } \frac{ds}{dr} = -\frac{1}{\sin 2\theta} = -\frac{a^2}{\sqrt{a^4 - r^4}}.$$

3. In the spiral of *Archimedes* whose equation is $r = a\theta$, it is required to prove that the tangent of the angle between the radius vector and a straight line through two points whose abscissæ are α, β is expressed by $\dfrac{(2\alpha - \beta)\sin(\alpha - \beta)}{(2\alpha - \beta)\cos(\alpha - \beta) - \alpha}$, the said radius vector belonging to the abscissa α.

4. In every curve referred to polar co-ordinates, it is required to establish the formulæ

$$\sin P = \frac{r\,d\theta}{ds}, \quad \cos P = \frac{dr}{ds} \text{ and } p = r^2 \frac{d\theta}{ds}.$$

5. The equation of the reciprocal spiral is $r = a\theta^{-1}$; then, if the notation of Article (193) be retained, $\tan P = -\dfrac{a}{r}$ and $p = \dfrac{ar}{\sqrt{a^2 + r^2}}$: required a proof.

6. If the equation of the spiral be $r = a\theta^m$; it is required to prove that $\tan P = \frac{1}{m}\left(\frac{r}{a}\right)^{\frac{1}{m}}$ and $p = \frac{r^{\frac{m+1}{m}}}{\sqrt{m^2 a^{\frac{2}{m}} + r^{\frac{2}{m}}}}$.

7. If the equation of the lemniscata be $r^2 = a^2 \cos 2\theta$; then will

$$\tan P = -\frac{r^2}{\sqrt{a^4 - r^4}} \text{ and } p = \frac{r^3}{a^2}:$$

required a proof.

8. In the cardioide whose equation is $r = a(1 + \cos\theta)$, prove that $\tan P = -\frac{r}{\sqrt{2ar - r^2}}$ and $p = -r\sqrt{\frac{r}{2a}}$.

9. If the spiral be defined by the equation $r = a \sin 2\theta$; then will $\tan P = \frac{1}{2}\frac{r}{\sqrt{a^2 - r^2}}$ and $p = \frac{r^2}{\sqrt{4a^2 - 3r^2}}$; required a proof.

10. The equation to a spiral being $r = a \sin m\theta$, it is required to prove that

$$\tan P = \frac{r}{m\sqrt{a^2 - r^2}} \text{ and } p = \frac{r^2}{\sqrt{m^2 a^2 - (m^2 - 1) r^2}}.$$

11. In the spiral defined by the equation $r^2 = a^2 \tan 2\theta$, it is required to prove that

$$\tan P = \frac{a^2 r^2}{a^4 + r^4} \text{ and } p = \frac{a^2 r^3}{\sqrt{a^8 + 3a^4 r^4 + r^8}}.$$

12. If the equation of a circle, of which the fixed axis is the chord of a quadrant, be $r = a(\cos\theta - \sin\theta)$; then will $\tan P = -\frac{r}{\sqrt{2a^2 - r^2}}$ and $p = -\frac{r^2}{a\sqrt{2}}$; required a proof.

13. The equation of an equilateral hyperbola from the centre being $r^2 = a^2 \sec 2\theta$, it is required to prove that $\tan P = \frac{a^2}{\sqrt{r^4 - a^4}}$ and $p = \frac{a^2}{r}$: also that $P = \frac{1}{2}\pi - 2\theta$ and $p = a\sqrt{\cos 2\theta}$.

14. The equation of a rectangular hyperbola referred to one of its asymptotes being $r^2 = 2a^2 \operatorname{cosec} 2\theta$; it is required to prove that $\tan P = -\dfrac{2a^2}{\sqrt{r^4 - 4a^4}}$ and $p = \dfrac{2a^2}{r}$: also that $P = \pi - 2\theta$ and $p = a\sqrt{2 \sin 2\theta}$.

15. The involute of a circle being defined by the equation $\theta = \dfrac{\sqrt{r^2 - a^2}}{a} - \sec^{-1}\dfrac{r}{a}$; it is required to prove that

$$\tan P = \frac{\sqrt{r^2 - a^2}}{a} \text{ and } p = \sqrt{r^2 - a^2}.$$

16. If the equation of the helicoid parabola be $(a-r)^2 = b^2\theta$: then will $\tan P = -\dfrac{2(ar - r^2)}{b^2}$ and $p = \dfrac{2r^2(a-r)}{\sqrt{(a-r)^2 r^2 + b^4}}$: required a proof.

17. The equation of the trisectrix being $r = a(2\cos\theta \pm 1)$; it is required to prove that $\tan P = -\dfrac{r}{\sqrt{3a^2 \pm 2ar - r^2}}$ and $p = \dfrac{r^2}{\sqrt{3a^2 \pm 2ar}}$.

18. If the equation of a polar curve be $r = a(\sec\theta - \tan\theta)$; prove that $\tan P = -\dfrac{2ar}{a^2 + r^2}$ and $p = \dfrac{2ar^2}{\sqrt{a^4 + 6a^2r^2 + r^4}}$.

19. In the elliptic spiral defined by the equation

$$\sqrt{a^2 - b^2}\,\theta = a\cos^{-1}\frac{\sqrt{a^2 - b^2}}{r},$$

it is required to prove that $\sin P = \dfrac{a}{\sqrt{b^2 + r^2}}$ and $p = \dfrac{ar}{\sqrt{b^2 + r^2}}$.

20. In the hyperbolic spiral defined by the equation

$$\sqrt{a^2 - b^2}\,\theta = b\log\frac{\sqrt{a^2 - b^2} + \sqrt{a^2 - b^2 - r^2}}{r},$$

it is required to prove that $\sin P = \dfrac{b}{\sqrt{a^2 - r^2}}$ and $p = \dfrac{br}{\sqrt{a^2 - r^2}}$.

21. In the spiral whose equation is

$$\theta = b \log \frac{\sqrt{a^2 - b^2 + r^2} + \sqrt{a^2 - b^2}}{r},$$

it is required to prove that $\sin P = \dfrac{b}{\sqrt{a^2 + r^2}}$ and $p = \dfrac{br}{\sqrt{a^2 + r^2}}$.

22. If a spiral be defined by $\sqrt{a^2 + b^2}\,\theta = a \sec^{-1} \dfrac{r}{\sqrt{a^2 + b^2}}$,

then will $\sin P = \dfrac{a}{\sqrt{r^2 - b^2}}$ and $p = \dfrac{ar}{\sqrt{r^2 - b^2}}$: required a proof.

23. If $(r - a)\,\theta^2 = r$ be the equation of a spiral, it is required to prove that

$$\sin P = \frac{a r^{\frac{1}{2}}}{\sqrt{a^2 r + 4\,(r - a)^3}} \text{ and } p = \frac{a r^{\frac{3}{2}}}{\sqrt{a^2 r + 4\,(r - a)^3}}.$$

24. Find the values of $\tan P$ and p in an ellipse and hyperbola referred to the centre, focus and vertex as a pole.

25. In any conic section, given the angle included between a tangent and the diameter through the point of contact, to find that point.

26. Required the same, when the angle included between the tangent and the distance from the focus is given in each of the conic sections.

27. The polar equation of a straight line being

$$r \sin (\alpha + \theta) = a \sin \alpha;$$

it is required to prove that $P = \pi - \alpha - \theta$ and $p = a \sin \alpha$.

28. Determine the values of $\tan P$ and p in the spiral defined by the equation $r = a \log \tan (45 + \frac{1}{2}\theta)$.

29. In the spiral whereof the equation is $(a^m + r^m)\,p^2 = r^{m+2}$,

it is required to prove that $\tan P = \left(\dfrac{r}{a}\right)^{\frac{m}{2}}$.

30. Find the values of $\tan P$ and p in the spiral defined by the equation $r = a\,(1 + 2 \sin \frac{1}{2}\theta)$.

31. Required the values of $\tan P$ and p in the curve whose polar equation is $r = a \sec\theta \tan^2\theta$.

32. One of the polar equations to a rectangular hyperbola being $r = a(\operatorname{cosec}\theta - \sec\theta)$; it is required to find the corresponding values of $\tan P$ and p.

33. In the rectangular hyperbola as defined by the equation $r\cos 2\theta = 2a\cos\theta$, it is required to find the values of $\tan P$ and p.

34. If the notation of Article (196) be retained, it is required to prove that in the spiral of *Archimedes* whose equation is $r = a\theta$, the polar subtangent $ST = \frac{r^2}{a}$ and the polar tangent $PT = \frac{r}{a}\sqrt{a^2 + r^2}$.

35. In the circle referred to a point in its circumference when $r = a\cos\theta$, it is required to prove that $ST = -\frac{r^2}{\sqrt{a^2 - r^2}}$ and $PT = \frac{ar}{\sqrt{a^2 - r^2}}$.

36. If a spiral be defined by the equation $r = a\sec\theta$; it is required to prove that

$$ST = \frac{r^2}{\sqrt{r^2 - a^2}} \text{ and } PT = r\sqrt{\frac{2r^2 - a^2}{r^2 - a^2}}.$$

37. If $r = a\theta^m$ be the equation of a spiral; it is required to prove that $ST = \frac{r\theta}{m}$ and $PT = \frac{r}{m}\sqrt{m^2 + \theta^2}$.

38. If a spiral be defined by the equation

$$\theta = \log\frac{a + \sqrt{a^2 - r^2}}{r};$$

it is required to shew that

$$ST = -\frac{ar}{\sqrt{a^2 - r^2}} \text{ and } PT = r\sqrt{\frac{2a^2 - r^2}{a^2 - r^2}}$$

39. If the equation $p = \dfrac{ar}{\sqrt{b^2 - r^2}}$ belong to a polar curve; then will $ST = \dfrac{ar}{\sqrt{a^2 - b^2 - r^2}}$ and $PT = r\sqrt{\dfrac{a^2 - r^2}{a^2 - b^2 - r^2}}$: required a proof.

40. Find the values of the polar subtangents and tangents in all the conic sections referred to the centres, foci and vertices: in the cissoid of *Diocles* and in the conchoid of *Nicomedes*.

41. If a line be drawn through the focus of an ellipse making an angle θ with the major axis, and tangents be drawn at the extremities of this line; it is required to prove that these tangents will include an angle ϕ, such that $(1-e^2)\tan\phi = 2e\sin\theta$.

42. If a and b be the semiaxes of an ellipse, and θ and ϕ the angles which any two conjugate diameters make with the major axis; prove that $a^2\tan\theta = b^2\cot\phi$.

43. A line drawn parallel to the focal distance of an ellipse or hyperbola, through the centre and meeting the tangent, is equal to the axis major: required a proof.

44. Prove that the tangent to any point of a conic section makes equal angles with the focal distances.

45. In the spiral defined by the equation $r = a\sin 2\theta$, if a straight line be drawn through any point parallel to the polar subtangent, it is required to shew that the length of it intercepted between the rectangular axes through the pole is invariable.

46. If from any point without a conic section, two tangents be drawn to the curve and two straight lines to the foci: it is required to prove that the angles between the tangents and straight lines will be equal.

47. If a be the radius of a circle and θ the angle at its centre, then may a tangent be drawn to it by means of the equation $r'\cos(\theta - \theta') = a$, if r' and θ' be its polar co-ordinates: required a proof.

48. If ϕ be the angle which a tangent to any point of the logarithmic spiral makes with the fixed axis; then may

this angle be found from the equation $\tan(\phi - \theta) \log a = 1$: required a proof.

49. If the notation of Article (199) be retained, it is required to prove that in the reciprocal spiral the polar subnormal $SG = -\frac{r^2}{a}$ and the polar normal $PG = \frac{r\sqrt{a^2 + r^2}}{a}$.

50. If the equation of a spiral be $r = a\theta^m$, then will $SG = ma^{\frac{1}{m}} r^{\frac{m-1}{m}}$, and $PG = r\sqrt{1 + m^2 \left(\frac{a}{r}\right)^{\frac{2}{m}}}$: required a proof.

51. In the spiral defined by the equation $r = \frac{2a}{e^{\theta} + e^{-\theta}}$, it is required to prove that

$$SG = -\frac{r}{a}\sqrt{a^2 - r^2} \text{ and } PG = \frac{r}{a}\sqrt{2a^2 - r^2}.$$

52. In the spiral whose equation is $r\cos^3\theta = a$, it is required to prove that

$$SG = \frac{3r\sqrt{r^{\frac{2}{3}} - a^{\frac{2}{3}}}}{a^{\frac{1}{3}}} \text{ and } PG = \frac{r\sqrt{9r^{\frac{2}{3}} - 8a^{\frac{2}{3}}}}{a^{\frac{1}{3}}}.$$

53. If a spiral be defined by the equation $r = a - b\theta^{\frac{1}{2}}$, then will $SG = -\frac{b^2}{2(a-r)}$ and $PG = \frac{\sqrt{4(ar - r^2)^2 + b^4}}{2(a-r)}$: required a proof.

54. In the involute of the circle whose radius is a, it is required to prove that $SG = \frac{ar}{\sqrt{r^2 - a^2}}$ and $PG = \frac{r^2}{\sqrt{r^2 - a^2}}$.

55. If the equation of a spiral be $r^2 = a^2\cos 2\theta$; it is required to prove that $SG = -\frac{\sqrt{a^4 - r^4}}{r}$ and $PG = \frac{a^2}{r}$.

56. If a spiral be defined by the equation $r = a\sec\theta$; it is required to prove that $SG = \frac{r}{a}\sqrt{r^2 - a^2}$ and $PG = \frac{r^2}{a}$.

57. The equation of the parabola being $r = a \operatorname{cosec}^2 \tfrac{1}{2}\theta$; it is required to prove that $SG = -r\sqrt{\dfrac{r-a}{a}}$ and $PG = r\sqrt{\dfrac{r}{a}}$.

58. In any conic section, if normals be drawn at the extremities of a chord passing through the focus so as to intersect each other; the sum of the squares of the parts of the normals between the curve and their intersection will have a given ratio to the square of the chord: required a proof.

59. The notation of Article (200) being retained, if the equation of a spiral be $r = a + b \sec\theta$, it is required to shew that $\angle LSX = 90^0$ and $ST = -b$.

60. If the equation of a spiral be $r^2 = a^2 \tan 2\theta$; it is required to prove that $\angle LSX = 45^0$ and $ST = 0$.

61. If a spiral be defined by the equation $r^2 = a^2 \sec 2\theta$; it is required to prove that $\angle LSX = 45^0$ and $ST = 0$.

62. In a spiral defined by the equation $r \cos\theta + a \cos 2\theta = 0$, it is required to prove that $\angle LSX = 90^0$ and $ST = a$.

63. If the equation of a polar curve be $r \cos 2\theta = 2a \cos\theta$; then will $\angle LSX = 45^0$ and $ST = \dfrac{a}{\sqrt{2}}$: required a proof.

64. The equation of a polar curve being $r \cos\theta = a \cos 2\theta$; it is required to prove that $\angle LSX = 90^0$ and $ST = -a$.

65. In the curve defined by the equation

$$r(\sin^3\theta + \cos^3\theta) = a \cos^2\theta,$$

it is required to prove that $\angle LSX = 135^0$ and $ST = -\dfrac{a}{3\sqrt{2}}$.

66. If a curve be defined by the equation $r^2 = a^2(\tan^2\theta - 1)$; then will $\angle LSX = 90^0$ and $ST = \pm a$: required a proof.

67. If the equation of a curve be

$$2r(\sin^3\theta + \cos^3\theta) = 3a \sin 2\theta;$$

then will $\angle LSX = 135^0$ and $ST = \dfrac{a}{\sqrt{2}}$: required a proof.

68. If a curve be defined by the equation

$$r^3(\cos^3\theta - \sin^3\theta) = a^3;$$

it is required to prove that $\angle LSX = 45^0$ and $ST = 0$.

69. In the polar curve defined by the equation

$$r^2\cos^2\theta\,(\tan\theta - 1) = a^2(\tan\theta + 1),$$

it is required to prove that $\angle LSX = 90^0$ and $ST = \pm a$ give two asymptotes: and $\angle LSX = 45^0$ and $ST = 0$, give a third.

70. The spiral being defined by the equation $r\cos\dfrac{\theta}{m} = a$;

it is required to prove that $\angle LSX = m\dfrac{\pi}{2}$ and $ST = ma$.

71. If the equation to a curve be $r\sin(m+1)\theta = a\sin m\theta$; it is required to prove that

$$\angle LSX = \frac{\pi}{m+1} \text{ and } ST = a\,\frac{\sin^2\dfrac{m\pi}{m+1}}{(m+1)\sin\dfrac{\pi}{m+1}}.$$

72. It is required to draw asymptotes to the polar curves defined by the following equations:

(1) $r^2\sin 2\theta = 2a^2$:

(2) $b\theta = a\sec^{-1}\dfrac{r}{b}$:

(3) $r = a(\operatorname{cosec}\theta - \sec\theta)$:

(4) $r(\cos\theta + \cot\theta) = 2a$:

(5) $r\sin 2\theta = a\sin\theta\sin 3\theta$:

(6) $r(e^\theta - 1) = a(e^\theta + 1)$:

(7) $r\sqrt{b^2\cos^2\theta - a^2\sin^2\theta} = ab$:

(8) $r(\theta^2 - 1) = a\theta^2$.

73. In the reciprocal spiral whose equation is $r = a\theta^{-1}$, the arc of the curve is always concave towards its pole: required a proof.

74. If $r = a^{\alpha-\theta}$ be the equation of the logarithmic spiral, it is required to prove that the curve is always concave towards its radius vector.

75. The equation of a circle being

$$r = b\cos\theta + \sqrt{a^2 - b^2\sin^2\theta};$$

it is required to find when the circumference is concave or convex towards its radius vector.

76. Find whether the spiral defined by the equation $r = a\theta^n$, is convex or concave towards its pole.

77. Determine whether the curve whose polar equation is $r = a(\operatorname{cosec}\theta - \sec\theta)$, is convex or concave towards its pole.

78. If the centre of the generating circle be the pole, the equation to a cycloid is $r^2 = a^2(1 + \theta^2 + 2\theta\sin\theta)$: find the direction of its curvature.

79. Find the direction of the curvature of the curve defined by the equation $r = a(\cos\theta - \sin\theta)$.

80. Determine whether the curve defined by the equation $r\cos 2\theta = a\cos\theta$, is convex or concave towards its pole.

81. Find the direction of the curvature of the curve whose polar equation is $r^2 = 2a^2\operatorname{cosec} 2\theta$.

82. Prove that the curve whose polar equation is

$$r\cos\theta = a\cos 2\theta,$$

is always concave towards the pole.

83. Shew from the equation $r\cos\theta = 2a\sin^2\theta$, that the arc of the cissoid of *Diocles* is always concave towards its radius vector.

84. If the notation of Article (205) be retained, prove that in the spiral of *Archimedes*, the radius of curvature $CP = \dfrac{(a^2+r^2)^{\frac{3}{2}}}{2a^2+r^2}$, and the chord of curvature $PV = \dfrac{2r(a^2+r^2)}{2a^2+r^2}$.

85. In the reciprocal spiral whose equation is $r = a\theta^{-1}$, it is required to prove that

$$CP = \frac{r(a^2+r^2)^{\frac{3}{2}}}{a^3} \text{ and } PV = \frac{2a(a^2+r^2)}{a^2}.$$

86. In the lemniscata of *Bernoulli* whose equation is $r^2 = a^2 \cos 2\theta$, it is required to prove that $CP = \frac{a^2}{3r}$ and $PV = \frac{2}{3}r$.

87. In the cardioide whose equation is $r = a(1 + \cos\theta)$, it is required to prove that $CP = \frac{2}{3}\sqrt{2ar}$ and $PV = \frac{4}{3}r$.

88. Determine the radius and chord of curvature of the curve whose polar equation is $r^2 = a^2(\tan^2\theta - 1)$.

89. Determine the radius and chord of curvature of the curve whose polar equation is $r = a \cot\theta \operatorname{cosec}\theta$.

90. Find the circle of curvature of the polar curve whose equation is $r^2 \cos^4\theta = a^2 \cos 2\theta$, and the magnitude of the chord passing through its pole.

91. Determine the radius and chord of curvature of the curve whose polar equation is $r^2(e^\theta - 1)^2 = 4a^2e^\theta$.

92. Find the magnitude of the radius and the position of the centre, of the circle of curvature of the curve whose equation is $r = a(\cos\theta - \sin\theta)$.

93. Determine the radius of curvature of the curve whose polar equation is $r \cos 2\theta = 2a \cos\theta$; and find where the centre is situated in terms of the co-ordinates.

94. Express the radius and chord of curvature of the curve defined by $r = a \sec\theta \tan^2\theta$, in terms of the polar co-ordinates.

95. Determine the magnitude and position of the circle of curvature of the curve whose equation is $r = a(1 + 2\sin\frac{1}{2}\theta)$, at a point where $\theta = 45^\circ$.

96. The equation of a curve being

$$r \sin(m + 1)\theta = a \sin m\theta,$$

it is required to find the radius and chord of curvature at a point where $\theta = 90^\circ$.

97. If the equation of a curve be $r = a(\cos\theta + 2\theta\sin\theta)$, it is required to find expressions for its radius and chord of curvature at any point.

98. Find expressions for the radius and chord of curvature of the curve defined by the equation $r(1 - e\sin 2\theta) = a\sin\theta$: and determine the points at which they are equal to α and β.

99. Prove that the radius of curvature at any point of a parabola referred to the focus is equal to half the polar subnormal.

100. If the equation of a curve be

$$\theta = \sqrt{\frac{2r - a}{a}} + \cos^{-1}\frac{r - a}{r},$$

it is required to prove that the curvature at any point varies inversely as the radius vector.

101. Apply the expressions investigated in Article (205) to determine the radii and chords of curvature of all the conic sections referred to their centres, foci and vertices.

102. If the equation of a spiral be $r = a^{\theta - \alpha}$; it is required to compare its radius of curvature with the normal, and its chord of curvature with the radius vector.

103. If u denote the inverse of the radius vector of a spiral, p the perpendicular upon the tangent, γ the radius of curvature and θ the spiral angle: then will

$$\frac{1}{p^2} = \left(\frac{du}{d\theta}\right)^2 + u^2 \text{ and } \frac{1}{\gamma} = p^3u^3\left(u + \frac{d^2u}{d\theta^2}\right):$$

required a proof.

104. Express the radius of curvature at any point of a parabola in terms of the angle which it makes with the fixed axis.

105. If the ordinary notation be retained, it is required to prove that $\frac{1}{\gamma} = \frac{d^2p}{dr^2}$, when the circle has contact of the third order with the spiral.

106. The equation to an epicycloid being

$$\theta = \frac{e}{a} \tan^{-1} \frac{r^2 - a^2}{e^2 - r^2} - \tan^{-1} \frac{e(r^2 - a^2)}{a(e^2 - r^2)},$$

it is required to prove that $p^2 = \frac{e^2(r^2 - a^2)}{e^2 - a^2}$, where $e = a + 2b$, if $a =$ the radius of the base and $b =$ the radius of the generating circle.

107. It is required to find the evolutes of the epicycloid and hypocycloid.

108. Find the evolute of the spiral defined by the equation $r = a^{\alpha - \theta}$.

109. If the equation of a spiral be $\sqrt{r^2 - a^2} = a\theta + a \sec^{-1} \frac{r}{a}$, it is required to find that of its evolute.

110. If $\theta \propto \frac{1}{r^m}$, it is required to prove that the polar subtangent at any point : the corresponding circular arc whose radius is r :: m : 1.

111. If $\theta \propto r$, the number of revolutions made by the radius vector varies as the square root of the polar subtangent: required a proof.

112. If a spiral be defined by the equation $r^2 \cos 2\theta = a^2$, it is required to prove that the sine of the angle at which the radius vector is inclined to the tangent $\propto \frac{1}{r^2}$.

113. Given the radius vector at any point of a parabola and the angle it makes with the curve, to find the position of the vertex and the magnitude of the *latus rectum*.

114. If $\frac{\alpha - \pi}{\pi} \log \frac{r}{a} = \log \sin \left(\frac{\alpha - \pi}{\pi} \theta\right)$ be the equation of a polar curve, it is required to shew that two tangents drawn to the curve at the extremities of a line passing through its pole will always include the same given angle.

115. In any spiral, if O be the point of intersection of perpendiculars to two consecutive radii vectores through their extremities, it is required to prove that $SP^2 = SO \,.\, Sy$.

116. In the focal distance SP of a parabola, SQ is taken equal to the rectangular ordinate MP: it is required to draw a tangent to the locus of the point Q.

117. If Q be a point so taken in the radius vector SP of a parabola that SQ is equal to the perpendicular upon the tangent from the focus: it is required to find the values of $\tan P$ and p in the locus of Q.

118. Prove that the spiral whose equation is

$$(\pi - 2\theta)\, r = 2a \cos\theta,$$

cuts off equal arcs from all circles passing through its pole and having their centres in its axis; and that the distances from the pole at which the spiral cuts the axis are as $\frac{1}{1}, \frac{1}{3}, \frac{1}{5}$, &c. the tangents of the angles in which it cuts it being as 1, 3, 5, &c.

119. Determine the asymptotic circle of the curve defined by the equation $\theta\sqrt{ar - r^2} = 1$.

120. In any polar curve, if r be considered the independent variable, it is required to prove that the radius of curvature is expressed by

$$CP = \frac{\left\{1 + \left(\frac{r\,d\theta}{dr}\right)^2\right\}^{\frac{3}{2}}}{r\frac{d^2\theta}{dr^2} + 2\frac{d\theta}{dr} + r^2\left(\frac{d\theta}{dr}\right)^3}.$$

121. If the arc s of the spiral be taken as the independent variable, it is required to shew that

$$CP = \frac{\frac{d\theta}{ds}}{r\left(\frac{d\theta}{ds}\right)^2 - \frac{d^2 r}{ds^2}}.$$

122. If neither θ nor r be considered the principal variable, it is required to prove that

$$CP = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 + r\frac{dr}{d\theta}\frac{d^2\theta}{d\theta^2} - r\frac{d^2r}{d\theta^2}},$$

$$\text{and } PV = \frac{r\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 + r\frac{dr}{d\theta}\frac{d^2\theta}{d\theta^2} - r\frac{d^2r}{d\theta^2}}.$$

123. It is required to apply the criterion deduced in Article (211) to ascertain whether the spiral whose equation is $r(e^\theta + e^{-\theta}) = 2a$, is concave or convex towards its pole.

124. If the tangent to a spiral make an angle ϕ with the rectangular ordinate, and p be the perpendicular let fall upon it from the origin: it is required to prove that

$$\pm\frac{ds}{d\phi} = p + \frac{d^2p}{d\phi^2},$$

where the upper or lower sign is to be used according as the curve is convex or concave towards its pole.

CHAP. X.

MISCELLANEOUS EXAMPLES.

By reference to Articles (213)–(251), the solutions of the following Theorems and Problems may be effected.

1. The equation of a curve being $y-1=(x-2)^3$, it is required to prove that it cuts the axis of x when $x=1$ and $\tan X=3$: and the axis of y when $y=-7$ and $\tan Y=-\frac{1}{6}$.

2. The curve whose equation is $ay^2=x^2(x-b)$, meets the axis of x when $x=0$ and $x=b$, and corresponding to them $\tan X=\pm\sqrt{\frac{b}{a}}$ and $\tan X=\infty$: required a proof.

3. If a curve be defined by the equation

$$x(a-x)=y(a+y),$$

it meets the axis of x when $x=0$ and $x=a$, so that $\tan X=\pm 1$: and it meets the axis of y when $y=0$ and $y=-a$, so that $\tan Y=\pm 1$: required a proof.

4. The curve whose equation is $\sqrt{xy}=a+x$, meets the axis of x when $x=-a$ and $\tan X=0$: required a proof.

5. If the equation of a curve be $a^3y=x^4-bx^3-b^2x^2$; then will it meet the axis of x when $x=0$ and $x=\frac{1}{2}b(1\pm\sqrt{5})$, $\tan X=0$ and $\tan X=\left(\frac{b}{a}\right)^3$; and the axis of y when $y=0$ and $\tan Y=\infty$: required a proof.

6. It is required to prove that the ellipse defined by the equation $y^2-2xy+3x^2+2y-4x-3=0$, meets the axis of x when $x=\frac{1}{9}(2\pm\sqrt{13})$ and $\tan X=\frac{1}{4}(5\mp\sqrt{13})$: also the axis of y when the values of y are 1 and -3, and $\tan Y=\frac{2}{3}$ and 2.

7. The parabola defined by the equation $y^2 - 4xy + 4x^2 - 8y + 3x - 2 = 0$, meets the axis of x when $x = \frac{-3 \pm \sqrt{41}}{8}$ and $\tan X = \frac{-41 \pm 13\sqrt{41}}{64}$: and the axis of y when $y = 4 \pm 3\sqrt{2}$ and $\tan Y = \frac{-96 \pm 78\sqrt{2}}{41}$: required a proof.

8. Determine where the curve whose equation is $y^3 - x^3 + a^3 = 0$, meets the co-ordinate axes, and the angles it makes with them.

9. Find where the curve whose equation is $(a - x)\,y^2 = (a + x)\,x^2$, intersects the axes of co-ordinates, and the angles in which it cuts them.

10. The curve whose equation is $ay^2(x - a) = x^4$, has an infinite ordinate when $x = a$, which is an asymptote to the curve: required a proof.

11. The curve defined by the equation $a^2xy = x^4 - b^4$, has an infinite ordinate through the origin, which is an asymptote to it: required a proof.

12. If $y\,(a + c + x) = b\,(c + x)$ be the equation of a curve, it is required to prove that when $x = -(a + c)$, the ordinate parallel to the axis of y is infinite, and becomes an asymptote; and when $y = b$, the ordinate parallel to the axis of x becomes infinite, and is then an asymptote.

13. The equation of a curve being $(x + a)\,y^2 = b^2(x + 2a)$, it is required to show that when $x = -a$, the ordinate parallel to the axis of y is infinite; and when $y = \pm b$, the ordinates parallel to the axis of x are infinite.

14. Find the positions of the infinite ordinates in the curves whose equations are $y = \tan x$ and $y = \sec x$.

15. The equation of an ellipse being $2y^2 - 4xy + 5x^2 - 3x = 0$, it is required to prove that the two values of y are equal to one another when $x = 0$ and when $x = 1$: that the two values of x are equal to each other when $y = \frac{1}{4}(2 \pm \sqrt{10})$; and that the corresponding ordinates are tangents to the curve.

16. In the curve whose equation is $y^4 - 2axy^2 + x^4 = 0$, it is required to prove that two values of y are equal when $x = 0$ and $x = a$; and that the corresponding ordinates are tangents to the curve.

17. In the curve whose equation is $y^4 - 96a^2y^2 + 100a^2x^2 - x^4 = 0$, the values of y become equal when $x = \pm 6a$ and $x = \pm 8a$, and the corresponding ordinates are tangents to the curve: prove this, and shew that the values of x can never be equal to one another.

18. If $y^2 = a^2 - x^2$ be the equation of a curve; then, when $x = 0$, $y = \pm a$, a positive and negative *maximum;* and when $y = 0$, $x = \pm a$, a positive and negative *maximum*: required a proof.

19. If $(y-a)^2 + (x-b)^2 = c^2$, then will the values of y be $a \pm c$, a *maximum* and a *minimum* when $x = b$: and the values of x will be $b \pm c$, a *maximum* and a *minimum,* when $y = a$: required a proof.

20. If the equation to a conic section be $y^2 - 2exy + x^2 - ay = 0$, it is required to prove that when $x = 0$, $y = 0$, a *minimum;* and when $x = \frac{ae}{1-e^2}$, $y = \frac{a}{1-e^2}$, a *maximum:* also, that when $y = \frac{a}{2(1+e)}$, $x = \frac{a}{2(1-e)}$, a *maximum;* and when $y = \frac{a}{2(1-e)}$, $x = -\frac{a}{2(1+e)}$, a *minimum* or negative *maximum.*

21. In the curve belonging to the equation $(x^2 + y^2)y = a(x+y)^2$, it is required to prove that when $x = 2a$, $y = 2a$, a *maximum* ordinate; and when $x = \infty$, $y = a$, which is neither a *maximum* nor a *minimum.*

22. In the curve whose equation is $x^4 - a^2x^2 + a^2y^2 = 0$, it is required to shew, that when $x = \pm\frac{a}{\sqrt{2}}$, $y = \pm\frac{1}{2}a$, which are *maxima;* and when $y = 0$, $x = \pm a$, which are also *maxima.*

23. If the equation of a curve be $(y-a)^2 = x(x-b)^2$, it is required to shew that when $x = \frac{1}{3}b$, and $y = a \pm \frac{2}{3\sqrt{3}} b^{\frac{3}{2}}$, the upper is a *maximum* and the lower a *minimum* ordinate.

24. Determine the *maximum* and *minimum* values of the co-ordinates of the curves defined by the following equations:

(1) $ax - ay = x^2 + y^2$: (2) $x^2y^2 = a^2(x^2 - y^2)$:
(3) $x^2y^2 = (x-9)(x-2)^2$: (4) $ay^3 = (x^2 - a^2)^2$:
(5) $y^2(a-x) = (a-2x)x^2$: (6) $a^2y^2 = x^2(x^2+y^2)$:
(7) $(x^2+y^2)^2 = a^2x^2 + b^2y^2$: (8) $(y^2 - x^2)^2 = ax^4 + bx^2$:
(9) $a^2y^2 = x^3(2a - x)$: (10) $a^2(x^2-y^2)^2 = (x^2+y^2)^3$:
(11) $y(x-2a) = x(x-a)$: (12) $(2ax - x^2)y^2 = (ax - x^2)^2$:
(13) $(2a-x)y^2 = x(a+b-x)^2$: (14) $(x^2+y^2)^2 = 4a^2xy$:
(15) $(y \pm x)^2 = 2ax - x^2$: (16) $x^5 + a^3xy = y^5$:
(17) $(a+x)^2y^2 = x^2(3a^2 - 2ax - x^2)$:
(18) $(a^2 - x^2)y = x\sqrt{2a^4 - 2a^2x^2 + x^4} + a^2x$.

25. Prove that the curve defined by the equation $y = ax + bx^2 - cx^3$, has a point of contrary flexure corresponding to $x = \frac{b}{3c}$ and $y = \frac{9abc + 2b^3}{27c^2}$.

26. The curve whose equation is $y = x^4 - 12x^3 + 48x^2 - 64x$, has points of inflexion corresponding to $x = 2$ and $y = -16$: also to $x = 4$ and $y = 0$: required a proof.

27. If the equation of a curve be $a^4y = 3x^5 - 35ax^4 + 140a^2x^3 - 240a^3x^2$, then corresponding to points of inflexion, the values of x are a, $2a$ and $4a$: required a proof.

28. In the curve whose equation is $y^3 = x^3 - a^3$, it is required to shew that when $x = 0$ and $y = -a$, also when $x = a$ and $y = 0$, there are points of contrary flexure.

29. The equation of a curve being $y = b + 2(x-a)^3$, it is required to prove that there is a point of inflexion when $x = a$; and to verify it by the substitution of $a \pm h$ for x.

30. If the equation of a curve be $ay = x\sqrt{2ax - x^2}$: then corresponding to $x=0$ and $y=0$, as also to $x = \frac{1}{2}a\,(3-\sqrt{3})$ and $y = \frac{1}{2}a\sqrt{6\sqrt{3}-9}$, there are points of contrary flexure: required a proof.

31. In the companion of the cycloid whose equations are $x = a\theta$ and $y = a\,(1 + \cos\theta)$, it is required to shew that there is a point of inflexion when $x = \frac{1}{2}\pi a$ and $y = a$.

32. The quadratrix of *Tschirnhausen* being defined by the equation $\frac{y}{a} = \sin\frac{\pi x}{2a}$, it is required to prove that there are points of contrary flexure corresponding to $x = 2ma$ and $y = a \sin m\pi = 0$.

33. The syntractrix is defined by the equation

$$x = a \log \frac{b + \sqrt{b^2 - y^2}}{y} - \sqrt{b^2 - y^2};$$

and it has points of inflexion corresponding to

$$x = a \log \frac{\sqrt{2a-b} + \sqrt{a-b}}{\sqrt{a}} - b\sqrt{\frac{a-b}{2a-b}} \text{ and } y = b\sqrt{\frac{a}{2a-b}}:$$

required a proof.

34. Of the Witch defined by the equation $xy^2 + a^2x = a^3$, the points whose co-ordinates are $x = \frac{3}{4}a$ and $y = \pm\frac{a}{\sqrt{3}}$, are points of inflexion: required a proof.

35. If a curve be defined by the equation $y = ae^{\cos x}$, then will its inflexions correspond to $\cos x = \frac{1}{2}(-1 \pm \sqrt{5})$ and $y = ae^{\frac{1}{2}(-1\pm\sqrt{5})}$: required a proof.

36. If $a \text{ vers } y = x$ be the equation of a curve, its inflexions correspond to $x = a$ and $y = (2m + 1)\frac{\pi}{2}$: required a proof.

37. If the equation of a curve be $y = a \text{ vers } 2x$, then will its points of inflexion belong to $x = (2m+1)\dfrac{\pi}{2}$ and $y = a$: required a proof.

38. It is required to ascertain the positions of the points of inflexion in the curves belonging to the following equations:

(1) $a^2 y = b^2 x - x^3$:
(2) $a^3 y = x^2 (a^2 - x^2)$:
(3) $x^4 = a^2 y^2 + x^2 y^2$:
(4) $(x^2 + y^2)^2 = a^2 x^2 + b^2 y^2$:
(5) $(a^2 + x^2)\, y = a^3$:
(6) $x y^2 = a^2 (a - x)$:
(7) $(a - x)\, y^2 = a^2 x$:
(8) $x y^2 = a\,(x - b)(x - c)$:
(9) $(a - x)\, 2 x y^2 = (2ax - x^2)^2$:
(10) $(a - x)\, y^2 = a^2 x (2a - x)$:
(11) $(x - a)\, y^2 = (2a - x)\, x^2$:
(12) $(a^2 - x^2)\, y^2 = (a^2 + x^2)\, x^2$:
(13) $\log y = x^{\frac{1}{3}}$:
(14) $y = x + \cos x$:
(15) $a^2 y^2 = x^4 - (b^2 + c^2)\, x^2 + b^2 c^2$:
(16) $a^3 y^3 + b^3 x^3 = a^3 b^3$:
(17) $(2a - x)(x^2 + y^2) = b^2 x$:
(18) $x^4 - a^2 x^2 + a^2 y^2 = 0$:
(19) $(2a - x)\, y^2 = x\,(x - \frac{1}{2}a)(x - \frac{3}{2}a)$.

39. In the curve whose equation is $ay^2 - x^3 + bx^2 = 0$, it is required to prove that the origin is a conjugate point.

40. If the equation of a curve be $(a - x)\, y^2 = x^3 - bx^2$, prove that the origin of co-ordinates is a conjugate point.

41. If the equation of a curve be $xy^2 = (x - 9)(x - 2)^2$, it is required to prove the existence of a conjugate point corresponding to $x = 2$ and $y = 0$.

42. The equation of a curve being $a\,(y - a)^2 = x^2 (x - a)$, it is required to prove that there is a conjugate point when $x = 0$ and $y = a$.

43. If the equation of a curve be $a\,(y - b)^2 = (x - c)^3$, it is required to prove the existence of a conjugate point when $x = c$ and $y = b$.

44. In the curve whose equation is $ay^2 = (b + x)^2 x$, it is required to shew that there is a conjugate point when $x = -b$ and $y = 0$.

45. The curve whose equation is $y = (x+2)\sqrt{x+1} + 3$, has a conjugate point corresponding to $x = -2$ and $y = 3$: required a proof.

46. In the curve defined by the equation

$$y + a = (a - b + x)\sqrt{x - b},$$

there is a conjugate point when $x = -(a-b)$ and $y = -a$: required a proof.

47. If the equation of a curve be $y + a = (x-b)\sqrt{x-2b}$, it is required to shew that when $x = b$ and $y = -a$, there is a conjugate point.

48. It is required to prove that the curve whose equation is $a^2(y-b)^2 = (2cx - x^2)(c+x)^2$, has a conjugate point corresponding to $x = -c$ and $y = b$.

49. If the equation of a curve be $y^2 - 1 = (x+1)\sqrt{x-1}$, there will be conjugate points corresponding to $x = -1$ and $y = \pm 1$: required a proof.

50. The curve defined by the equation

$$(y-3)^6 = (x-2)^2(1-x)^3,$$

has a conjugate point when $x = 1$ and $y = 3$: required a proof.

51. The origin of the co-ordinates in the curve whose equation is $(y^2 - \frac{1}{2}a^2)^2 = \frac{1}{4}a^4 + 2a^2x^2 - 3a^2xy$, is a conjugate point: required a proof.

52. It is required to prove that the curve defined by the equation $a(y^2 - b^2) = (x^2 - c^2)\sqrt{c^2 - 2x^2}$, has four conjugate points corresponding to $x = \pm c$ and $y = \pm b$.

53. It is required to determine the positions of the conjugate points in the curves defined by the following equations:

(1) $(x^2 - a^2)^2 + (y^2 - b^2)^2 = 0$:

(2) $a(y-a)^2 = x^2(x-b)$:

(3) $a^3(y+a)^2 = (x-2b)(x-b)^4$:

(4) $(x+a)y^2 = (x-b)x^2$:

(5) $a^5(y-2a)^2 = (x-3b)(x-b)^6$:

(6) $x(x^2+y^2) = 3x - 2$:

(7) $xy^2 + x^3 = 7x^2 - 15x + 9$:

(8) $x^4 - ax^2y - ay^3 + a^2y^2 = 0$.

54. In the curve whose equation is $y^2 = a^2 + ax$, it is required to prove that the curvature is a *maximum* when $x = -a$ and $y = 0$.

55. In the logarithmic curve whose equation is $y = a^x$, it is required to prove that the curvature is a *maximum* when $y = a^x = \frac{1}{\sqrt{2}\log a}$.

56. It is required to prove that the maximum curvature of the curve, whose equation is $a^3 y = x(b^3 - x^3)$, takes place when $x = \frac{b}{\sqrt[3]{4}}$ and $y = \frac{3b^4}{4^{\frac{4}{3}} a^3}$.

57. The equation of a cycloid being

$$y = a \operatorname{vers}^{-1}\frac{x}{a} + \sqrt{2ax - x^2},$$

it is required to prove that the curvature is a *minimum* when $x = 0$, $y = 0$; and a *maximum* when $x = 2a$, $y = \pi a$.

58. In the catenary whose equation is $2\frac{x}{a} = e^{\frac{y}{a}} + e^{-\frac{y}{a}}$, it is required to shew that when $x = a$ and $y = 0$, the radius of curvature $= a$ a *minimum*.

59. Of the curve defined by the equation

$$\sqrt{a}\,(y - b) = \sqrt{x}\,(x - a),$$

it is required to prove that the *maximum* curvature corresponds to $x = 0$ and $y = b$.

60. If the equation of a curve be $(a - x)\,y^2 = (a + x)\,x^2$, it is required to prove that the curvature is a *maximum* when $x = -a$ and $y = 0$.

61. The equation of a curve being

$$a^2 (y - a)^2 = (2ax - x^2)(x - a)^2,$$

it is required to shew that when $x = 0$ and $y = a$, the curvature is a *maximum:* when $x = a$ and $y = a$, the curvature is a *minimum:* when $x = 2a$ and $y = a$, the curvature is a *maximum.*

62. Determine the points of *maximum* and *minimum* curvature in the curves defined by the following equations:

(1) $y^2 = 2(1 - x^2)$: (2) $4y^2 + x^2 - 4x = 0$:

(3) $a^2 y^2 = b^2 x(x - 2a)$: (4) $a(x - y) = xy$:

(5) $y - x + x^3 = 0$: (6) $\sqrt{y} = \sqrt{a} \pm \sqrt{x}$:

(7) $(x^2 + y^2)^2 = a^2 xy$: (8) $by^2 = ax - by$:

(9) $(e^x - 1)\, e^y = e^x + 1$: (10) $\sqrt{xy} = a + x$:

(11) $a^3 y = x^4 - bx^3 - b^2 x^2$: (12) $(a^2 + x^2)\, y = a^2 x$:

(13) $(2a - x)\, y^2 = x\,(a + b - x)^2$: (14) $a^2 x^2 y^2 = (x^2 + y^2)^3$.

63. In each of the examples just given, prove that the order of contact subsisting between the curve and its circle of curvature at the points there determined exceeds the second.

64. Prove that the curve whose equation is

$$x^4 - a^2 x^2 + a^2 y^2 = 0,$$

has a double point corresponding to $x = 0$ and $y = 0$: and find the directions of its arcs.

65. The equation of a curve being $(a - x)\, y^2 = x\,(b - x)^2$, it is required to prove that there exists a double point corresponding to $x = b$ and $y = 0$: and to find also the directions of its arcs there.

66. Prove that the curve defined by the equation

$$(a^2 - x^2)\, y^2 = x^4,$$

has a double point at the origin of co-ordinates: and find the directions of its branches.

67. The curve whose equation is

$$a^2 (y - a)^2 = (2ax - x^2)\,(x - a)^2,$$

has a double point corresponding to $x = a$ and $y = a$: required a proof, and find the directions of the arcs which form it.

68. In the curve whose equation is $(a^2 - x^2)\, y^2 = (a^2 + x^2)\, x^2$, there is a double point at the origin: prove this, and find the directions of the branches.

69. A curve being defined by the equation

$$y - a = (x - b)\sqrt{x - c},$$

it is required to prove that it has a double point when $x = b$ and $y = a$: and to find the directions of its branches.

70. If the equation of a curve be $(y-2)^2=(x-1)^2 x$, it is required to prove that it has a double point corresponding to $x=1$ and $y=2$: and to find the directions of the arcs forming it.

71. In the curve whose equation is $ay^2=x(a\pm x)^2$, it is required to shew that a double point corresponds to

$$x=\mp a \text{ and } y=0:$$

and to find the directions of its branches.

72. The equation of a curve being $xy^2=a(x-b)^2$, it is required to prove the existence of a double point corresponding to $x=b$ and $y=0$: and to find the directions of its branches.

73. The curve whose equation is $y^4-a^2y^2+2a^2x^2-x^4=0$, has a double point at the origin of co-ordinates: prove this, and find the directions of its branches.

74. If the equation of a curve be

$$y^4-8y^3-12xy^2+16y^2+48xy+4x^2-64x=0,$$

there will be a double point corresponding to $x=2$ and $y=2$: prove this, and find the directions of the branches.

75. The equation of a curve being

$$y^4+x^4-2a^2y^2-2b^2x^2+b^4=0,$$

it is required to shew that the curve has two double points corresponding to $x=\pm b$ and $y=0$: and to find the directions of the branches.

76. The curve whose equation is

$$y^4+x^4-2a^2y^2-2a^2x^2+a^4=0,$$

has two double points when $x=0$ and $y=\pm a$: and also two double points when $x=\pm a$ and $y=0$: required a proof, and find the directions of the arcs forming them.

77. The curve defined by the equation

$$xy^2+x^3-(a+2b)x^2+(2ab+b^2)x-ab^2=0,$$

has a double point corresponding to $x=b$ and $y=0$: prove this and find the directions of the branches.

78. The equation of a curve is

$$x^4-2^{\frac{3}{2}}ax^3+2a^2x^2-a^2y^2-ay^3=0;$$

it is required to shew that there are double points corresponding to $x=0$, $y=0$, and $x=a\sqrt{2}$, $y=0$: and to find the directions of the arcs by which they are formed.

79. The curve whose equation is $(x^2-a^2)^2=ay^2(2y+3a)$ has double points corresponding to $x=a$, $y=0$: $x=-a$, $y=0$ and $x=0$, $y=-a$: required a proof, and find the directions of the arcs at those points.

80. Prove that the curve defined by the equation

$$ay^3-bx^2y+x^4=0,$$

has a triple point at the origin of the co-ordinates, and determine the direction of its arcs.

81. The curve defined by the equation

$$(y^2+ax)^2=x^2(a^2+2ax-x^2),$$

has a triple point at the origin of its co-ordinates: prove this, and find the directions of its arcs there.

82. If the equation of a curve be $(y^2-x^2)^2=ax^3-2bxy^2$, it is required to shew that the origin is a triple point, and to find the directions in which the curve proceeds from that point.

83. Of the curve defined by the equation

$$y^4-2x^2y^2+x^4+2axy^2-5ax^3=0,$$

the origin is a triple point: required a proof, and determine the directions of its branches there.

84. In the curve whose equation is

$$y^4+x^4-4ay^3+2ay^2x+2ax^3+8a^2y^2-4a^2xy-8a^3y+2a^4=0,$$

it is required to prove the existence of a triple point when $x=-a$ and $y=a$.

85. In the curve whose equation is

$$x^4-ay^3+2ax^2y+4ax^3+3a^2y^2+4a^2xy+4a^2x^2-a^3y=0,$$

it is required to prove the existence of a triple point corresponding to $x=-a$ and $y=a$.

86. It is required to prove that the origin in the curve whose equation is $(x^2+y^2)^3 = a^2(x^2-y^2)^2$, is a quadruple point: and to find the directions of the arcs meeting there.

87. In the curve whose equation is $y-a=(x-b)^2\sqrt{x-c}$, it is required to prove that two branches of the curve have contact of the first order corresponding to $x=b$ and $y=a$.

88. In the curve defined by the equation

$$y-a=(x-b)^3\sqrt{x-c},$$

there will subsist a contact of the second order between two branches of the curve corresponding to $x=b$ and $y=a$: required a proof.

89. Determine the positions and natures of the multiple points in the curves defined by the following equations:

(1) $(x^2+y^2)^2=4axy$:

(2) $y^2(2ax-x^2)=(ax-x^2)^2$:

(3) $(a+x)^2y^2=x^2(3a^2-2ax-x^2)$:

(4) $(1-x)y^2=x(2-x)^2$:

(5) $y^2(2a-x)=x(b^2-2ax+x^2)$:

(6) $y^2(a^2-x^2)=x^2(b-x)^2$:

(7) $y^3-ax^2-x^3=0$:

(8) $a(b+y)^2=x^2(c+x)$:

(9) $(x^2+y^2)(2a-x)=b^2x$:

(10) $a^2(x^2-y^2)=x^2y^2$:

(11) $ay^2-x^3-bx^2=0$:

(12) $ay^3+x^3y-bx^3=0$:

(13) $ay^2-2aby=x^3+cx^2-ab^2$:

(14) $2x(x^2+y^2)=a(x^2-y^2)$:

(15) $(y^2-x^2)^2+6axy^2-7ax^3-4a^2y^2+18a^2x^2-20a^3x+8a^4=0$:

(16) $y^4+x^2y^2-8ay^3-4axy^2-4ax^2y+19a^2y^2+16a^2xy+5a^2x^2$
$-12a^3y-14a^3x-3a^4=0$.

90. Prove that the semicubical parabola whose equation is $ay^2=x^3$, has a cusp of the first species at the origin, the tangent of which is the axis of x.

91. The curve defined by the equation $a(y-a)^2=(a-x)^3$, has a cusp of the first species corresponding to $x=a$ and $y=a$: required a proof.

92. It is required to prove that the curve whose equation is $a(x-y)^2=(x+y)^3$, has a cusp of the first species at the origin, and to find the direction of its rectilineal tangent.

93. The curve defined by the equation

$$a(x+y-1)^2=(x-y+3)^3,$$

has a cusp of the first species corresponding to $x=-1$ and $y=2$: prove this, and find the direction of its rectilineal tangent.

94. The equation of a curve being

$$a(x-y+1)^4=(x+y-5)^5,$$

it is required to prove that there exists a cusp of the first species when $x=2$ and $y=3$: and to find the direction of its rectilineal tangent.

95. It is required to prove that the curve whose equation is $x(y-1)^2=(2-x)^3$, has a cusp corresponding to $x=2$ and $y=1$: and to find the direction of its rectilineal tangent.

96. Prove that the curve whose equation is

$$(x^2+y^2)(x-y)=a(x+y)^2,$$

has a cusp of the first species at the origin of co-ordinates: and find the direction of its rectilineal tangent.

97. In the curve whose equation is

$$a^3(2y+x)^2=2(2b-x)^5,$$

it is required to prove the existence of a cusp of the first species corresponding to $x=2b$ and $y=-b$; and to find the direction of its rectilineal tangent.

98. If $y=a\cos^{-1}\left(\dfrac{x-a}{a}\right)+\sqrt{2ax-x^2}$ be the equation of a curve: prove the existence of a cusp of the first species corresponding to $x=0$ and $y=\pi a$: and find the direction of its rectilineal tangent.

99. The equation of a curve being $(y-x-a)^2=(x-a)^9$, it is required to prove that there is a cusp of the first species when $x=a$ and $y=2a$: and to find the direction of its rectilineal tangent.

100. The equation of a curve being

$$y = \log\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2},$$

it is required to shew that there is a cusp of the first species corresponding to $x=1$ and $y=0$: and to find its tangent.

101. The equation of a curve being $a^4y^2=(x^2-a^2)^3$, it is required to prove the existence of two cusps of the first species corresponding to $x=\pm a$ and $y=0$: and to find the directions of their tangents.

102. The curve defined by the equation

$$a^2y^2-2ax^2y=3x^4+bx^3,$$

has two cusps of the first species corresponding to $x=0$, $y=0$, and $x=-\frac{1}{4}b$, $y=\frac{b^2}{16a}$; required a proof: and find the positions of their tangents.

103. The equation of a curve being $4ay^3=(x^2-a^2)^2$, it is required to prove the existence of two cusps of the first species corresponding to $x=\pm a$ and $y=0$: and to find the directions of their tangents.

104. In the curve defined by the equation

$$y-a=bx^2+(x-1)^{\frac{5}{2}},$$

it is required to prove the existence of a cusp of the second species corresponding to $x=1$ and $y=a+b$: and to find the position of its rectilineal tangent.

105. If the equation of a curve be

$$(2x+y-x^2)^2=(x-1)^5,$$

it is required to prove the existence of a cusp of the second species when $x=1$ and $y=-1$: and to find the direction of its tangent.

106. A curve being defined by the equation

$$y-2=x\left(1+x+x^{\frac{3}{2}}\right),$$

it is required to prove that there is a cusp of the second species belonging to $x=0$ and $y=2$: and to find the direction of its tangent.

107. If a curve be defined by the equation

$$y+x+1=x^2-2x^{\frac{5}{2}},$$

there will be a cusp of the second species corresponding to $x=0$ and $y=-1$: prove this, and find the direction of its tangent.

108. In the curve defined by the equation

$$x^4-ax^2y-axy^2+\tfrac{1}{4}a^2y^2=0,$$

it is required to prove the existence of a cusp of the second species at the origin of co-ordinates: and to find the direction of its tangent.

109. Determine the positions and species of the cusps belonging to the curves defined by the following equations:

(1) $(y-x)^2=x^3$:

(2) $(y-a)^2=(x-b)^5$:

(3) $(y\mp b\sqrt{x})^3=a^3x$:

(4) $(y-a)^3=(x-b)^2$:

(5) $(2y+x+1)^2=4(1-x)^5$:

(6) $(y-x-a)^2=(x-a)^3$:

(7) $(y-x-a)^2=(ax^2+bx^{\frac{5}{2}})^2$:

(8) $a(y-x)^2=(x-1)^3$:

(9) $(y-2)^3=(x-1)^2$:

(10) $(y-a)^5=(x-b)^2$:

(11) $y-a=b(x-a)^{\frac{1}{3}}+c(x-a)^{\frac{1}{4}}$:

(12) $y^3=ax^2+x^3$:

(13) $x^2(y^2+1)=2a^2+(x-b)^2$:

(14) $x^{\frac{2}{3}}-y^{\frac{2}{3}}=(2a)^{\frac{2}{3}}$:

(15) $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$:

(16) $(y+x)^{\frac{2}{3}}-(y-x)^{\frac{2}{3}}=1$:

(17) $xy^2+2a^2y-ax^2-3a^2x-3a^3=0$:

(18) $x^4+x^3y-x^2y^2-2ax^2y+axy^2+a^2y^2=0.$

110. If the equation of a curve be

$$x^4 + x^3y - x^2y^2 + 2ax^2y + axy^2 + a^2y^2 = 0,$$

it is required to prove the existence of a point of complete *Embrassement* at the origin. See the Diagram of page (326).

111. In the curve defined by the equation

$$a^3y^2 - 2abx^2y = x^5,$$

the origin of co-ordinates is a point both of inflexion and osculation, commonly called a point of *Osculinflexion*: required a proof.

112. The curve whose equation is $x^5 + bx^4 = a^3y^2$, has at the origin of the co-ordinates a point of *Rebroussement* with a double inflexion; required a proof.

113. The curve defined by the equation

$$c^6y^2 = (a + b)\,c^3x^3y - abx^6 + c^3x^3y^2,$$

has a multiple point at the origin of co-ordinates: find its nature and circumstances.

114. The curve whose equation is

$$a^2y^3 - a^2by^2 + a^2x^2y + x^5 = 0,$$

has a point of *Osculinflexion* at the origin: required a proof.

115. In the curve defined by the equation $y^5 + ax^4 = b^2xy^2$, there is at the origin of co-ordinates a point of inflexion with a cusp of the first species: required a proof.

116. The curve defined by the equation $y^4 \pm x^4 + 2bx^2y = 0$, has at the origin of co-ordinates a triple point with a cusp of the first species: required a proof.

117. The curve whose equation is

$$x^5 - ay^4 + 2bx^3y + b^2xy^2 = 0,$$

has at the origin of co-ordinates a triple point with a cusp of the second species: required a proof.

118. The curve whose equation is

$$x^5 - ay^4 + 2bx^3y + \tfrac{1}{2}b^2xy^2 = 0,$$

has at the origin of co-ordinates a triple point with a point of *Embrassement:* required a proof.

119. The curve defined by the equation

$$x^6 + a^2y^4 + a^2x^3y + a^3xy^2 = 0,$$

has at the origin of co-ordinates a triple point with a point of *Embrassement:* required a proof.

120. The curve whose equation is $a^2y^3 - bx^3y + x^5 = 0$, has at the origin of co-ordinates a triple point with a point of *Rebroussement:* required a proof.

121. In the curve whose equation is $x^6 + 2a^2x^3y - b^3y^3 = 0$, there is at the origin of co-ordinates a triple point with a point of *Rebroussement:* required a proof.

122. In the curve whose equation is

$$(y - a - x)^4 = a(x - b)^3,$$

it is required to find the nature of the point corresponding to $x = b$.

123. Find the nature of the point corresponding to $x = 1$ and $y = -1$, in the curve whose equation is

$$(xy + 1)^2 = (2 - x)(x - 1)^3.$$

124. If a curve be defined by the equation

$$y - a = (x - b)^{\frac{1}{3}} + (x - b)^{\frac{3}{4}},$$

it is required to find the nature of the point belonging to $x = b$.

125. Find the nature of the point corresponding to $x = -a$ in the curve whose equation is

$$y - 2a = x + b(x + a)^2 + c(x + a)^{\frac{5}{2}}.$$

126. Required the magnitudes and positions of the greatest and least radii vectores in the curve defined by the polar equation $r = \dfrac{a}{1 + e \cos \theta}$.

127. Find the positions and magnitudes of the greatest and least radii vectores in the curve whose equation is

$$r^2 = a^2 \sin 2\theta.$$

128. Determine the *maximum* or *minimum* radius vector in the curve whose polar equation is $r \cos \theta = a (1 - \cos \theta)$.

129. Find the magnitude and position of the *maximum* or *minimum* radius vector of the Lituus, whose equation is

$$r^2 \theta = a^2.$$

130. Determine whether the spiral whose equation is

$$r (e^\theta + e^{-\theta}) = 2a,$$

admits of a *maximum* or *minimum* radius vector, and decide which it is.

131. Find the singular points of the spiral whose equation is $r = a \sin 2\theta$.

132. Determine the singular points of the polar curves whose equations are $r^2 = a^2 \cos^2 \theta \pm b^2 \sin^2 \theta$.

133. Find the positions and natures of the singular points in the curve whose polar equation is $r = a (1 + \cos \theta)$.

134. In the polar curve defined by the equation

$$2r (\sin^3 \theta + \cos^3 \theta) = 3a \sin 2\theta,$$

it is required to find the positions and natures of the singular points.

CHAP. XI.

MISCELLANEOUS EXAMPLES.

1. It is required to trace or describe the common parabola whose equation is $y^2 = 4ax$. See the second Diagram of page (304) with the axes of x and y interchanged.

2. Trace the parabolic curve whose equation is $ay^3 = x^4$. See the second Diagram of page (304).

3. Determine the form of the curve defined by the equation $ay^2 = x^3$. See the Diagram of page (338).

4. Required the form of the curve whose equation is $a^2y = x^3$. See the Diagram of page (303).

5. Trace the curve defined by the equation $y^3 = a^2x$. See the Diagram of page (303) reversed.

6. Trace the curve whose equation is $(y - a)^2 = 4a(x - a)$. See the second Diagram of page (304) with a new origin and the axes interchanged.

7. Find the form of the figure belonging to the equation $a(y + b)^3 = (x - a)^4$. See the Diagram just referred to.

8. The equation of a curve being $a(y - b)^2 = (x - c)^3$, find its figure. See the Diagram of page (338) with a new origin and parallel axes.

9. What is the form of the curve defined by the equation $a^2(y - b) = (x + a)^3$? See the Diagram of page (303) with a new origin and parallel axes.

10. Trace the curve whose equation is $x^4 - bx^3 = a^3y$. See the Diagram of page (346) inverted.

11. Find the figure belonging to the equation

$$x^4 + a^3 y + b^3 x = 0.$$

See the Diagram, last referred to, reversed.

12. Trace the curve defined by the equation

$$4y = x^3 - 3x^2 - 4x + 12.$$

See the first Diagram of page (302).

13. Find the form of the curve whose equation is

$$y = x^3 + 2x^2 - x - 2.$$

See the Diagram, just referred to, reversed.

14. Trace the curve defined by the equation

$$y + 1 = x(1 + 4x - 4x^2).$$

See the Diagram, last referred to, inverted.

15. Determine the figure of the curve whose equation is $a^2 y + cx^2 - x^3 = 0$. See the second Diagram of page (302).

16. If the equation of a curve be $y^3 - ay^2 = b^2 x$, find its figure. See the Diagram, last referred to, with its axes interchanged.

17. The equation of a curve being $4(y-1) = x^2(x^2 - 5)$, it is required to find its figure. See the first Diagram of page (304).

18. If the equation of a curve be $y^4 - 13y^2 = 36(x-1)$, it is required to trace it. See the first Diagram of page (304) with interchanged axes.

19. The equation of a curve being $9(y-x) = x^5 - 10x^3$, it is required to describe it. See the first Diagram of page (306).

20. The equation of a curve being $4(2y-x) = x^3(x^2-5)$, it is required to determine its figure. See the Diagram, last referred to, inverted.

21. If the equation of a curve be

$$6(2y+1) = x^2(x^4 - 6x^2 + 11),$$

it is required to find its figure. See the Diagram of page (307).

22. The equation of a curve being $y^6 - 9y^4 + 26y^2 = 24(1-x)$, it is required to trace it. See the Diagram, just referred to, inverted.

23. Trace the curve defined by the equation $x^2y = a^2(x-y)$. See the Diagram of page (287).

24. Describe the ellipse whose equation is

$$y^2 - 2xy + 3x^2 - 10x + 12 = 0.$$

See the Diagram of page (291).

25. Trace the rectangular hyperbola whose equation is $xy = a(x+y)$. See the first Diagram of page (289) with a new origin and parallel axes.

26. The equation of a curve being $y^2(1+x) = x^2(1-x)$, it is required to trace it. See the second Diagram of page (289) reversed.

27. If the equation of a curve be $x^4 - a^2x^2 + a^2y^2 = 0$, it is required to trace it. See the Diagram of page (294).

28. The equation of a curve being $(y^2+x^2)^2 = 2a^2(y^2-x^2)$, it is required to trace it. See the Diagram, just referred to, with axes interchanged.

29. Trace the curve whose equation is $(y-2)^2 = x(x-1)^2$. See the Diagram of page (317).

30. Trace the curve defined by the equation

$$a(y-b)^2 = (x-2a)(x-a)^2.$$

See the Diagram just referred to.

31. Find the form of the curve whose equation is

$$a^3y + x(b+x)^3 = 0.$$

See the Diagram of page (348) inverted.

32. Describe the curve whose equation is $a^3y = x(b+x)^3$. See the Diagram, just referred to, reversed.

33. Trace the curve belonging to the equation

$$a^3x = y(y-b)^3.$$

See the Diagram, last referred to, with its axes interchanged.

34. If a curve be defined by the equation $a^3y = x^2(b + x)^2$, it is required to trace it. See the Diagram of page (349) reversed.

35. Trace the curve whose equation is $a^3y + (x^2 + bx)^2 = 0$. See the Diagram, last referred to, inverted.

36. The equation of a curve being $y^3 - x^3 = a^3$, it is required to trace it. See the Diagram of page (350) reversed.

37. If the equation of a curve be $x^3 + y^3 + a^3 = 0$, it is required to trace it. See the Diagram, just referred to, inverted.

38. The equation of a curve being

$$(x + 2)\,y + (x + 1)\,(x + 3) = 0,$$

it is required to describe it. See the Diagram of page (352) reversed.

39. If the equation of a curve be $(x + 2)\,y = (x + 1)\,(x + 3)$, it is required to trace it. See the Diagram, just referred to, inverted.

40. A curve being defined by the equation

$$(1 - x^2)\,y = x\,(1 + x^2),$$

it is required to trace it. See the Diagram of page (353) reversed.

41. If a curve be defined by the equation $y^3 = x^3 - a^2x$, it is required to trace it. See the Diagram of page (354) reversed.

42. Trace the curve whose equation is

$$y - 1 = (x - 1)\sqrt{2x - x^2}.$$

See the Diagram of page (357).

43. Trace the curve whose equation is $(x^2 + y^2)^2 = a^2xy$. See the Diagram of page (294) with its axes changed through half a right angle.

44. If the equation of a curve be $(1 + x)\,y = (1 - x)\,x$, it it required to trace it. See the first Diagram of page (364) reversed.

45. Trace the curve whose equation is $(x+1)\,y=(x-1)\,x$. See the Diagram, just referred to, inverted.

46. The equation of a curve being $y^3=x^3+ax^2$, it is required to trace it. See the second Diagram of page (364) reversed.

47. If the equation of a curve be $y^3+x^3+ax^2=0$, it is required to trace it. See the Diagram, just referred to, inverted.

48. A curve being defined by the equation

$$y^4+(a^2+x^2)\,y^2=a^2x^2,$$

it is required to trace it. See the first Diagram of page (365) with its axes interchanged.

49. Trace the curve defined by the equation

$$4a^2xy=(x^2+y^2)\,(x+y)^2.$$

See the Diagram, just referred to, with its axes changed through half a right angle.

50. Trace the curve whose equation is

$$(2-x)\,y^2=x\,(x-\tfrac{1}{2})\,(x-\tfrac{3}{2}).$$

See the first Diagram of page (366).

51. Trace the circle whose equation is $(y-a)^2=a^2-x^2$, finding the position of its centre and the magnitude of its radius.

52. Describe the equilateral hyperbola as defined by the equation $(y-a)^2=x^2-a^2$.

53. Trace the rectangular hyperbola as defined by the equation $x\,(a-y)=y$.

54. It is required to trace the circle as defined by the equation $x^2+y^2=ax+by$.

55. Trace the common parabola by means of the equations $\sqrt{y}\pm\sqrt{x}=\sqrt{a}$.

56. Trace the common parabola whose equation is

$$(y-x)^2=3y.$$

57. Trace the ellipse as defined by the equation

$$y^2-2xy+3x^2+2y-4x-3=0.$$

58. Describe the hyperbola belonging to the equation $y^2 - 6xy + x^2 + 2y - 6x + 5 = 0$.

59. Trace the curve defined by the equation

$$x^4 y^4 - a^4 y^4 + b^8 = 0.$$

60. Trace the Cissoid of *Diocles* as defined by the equation $(x - y)(x^2 + y^2) = a(x + y)^2$.

61. Trace the curve defined by the equation

$$a^2 y^2 + x^2 y^2 = a^4.$$

62. Let it be required to describe the curves defined by the following equations:

(1) $y^2 (a - x) = x^3$:
(2) $xy^2 + a^2 x = a^3$:
(3) $ay^2 = x^2 (b + x)$:
(4) $x^4 = a^2 y^2 + x^2 y^2$:
(5) $x^4 + by^3 = ax^2 y$:
(6) $ay^2 = x^2 (x - b)$:
(7) $(y \pm x)^2 = x (2a - x)$:
(8) $(a^2 - x^2) y^2 = x^4$:
(9) $x (x^2 + y^2) = a (x^2 - y^2)$:
(10) $(2a - x) y^2 = x (a - x)^2$
(11) $(a^2 - x^2) y^2 = (a - x)^2 x^2$:
(12) $(a + x)^2 y^2 = (a^2 + x^2) x^2$:
(13) $(a - x)^2 (x^2 + y^2) = a^2 y^2$:
(14) $ay^2 = (a^2 + x^2) x$:
(15) $x^2 y^2 = (a + x)(a - x)^2$
(16) $(a^2 - x^2) y^2 = (2a^2 - x^2)^2$
(17) $a^2 y - x^2 y = a^3$:
(18) $(x^2 - 3x + 2) y = x - 3$.
(19) $(e^x - 1) y = 2e^x$:
(20) $(e^x - 1) y = e^x + 1$:
(21) $ay^2 - bxy = x^2 (x - b)$:
(22) $x^4 + 2ax^2 y = ay^3$:
(23) $x^4 + y^4 = 2axy^2$:
(24) $(x - a) y^2 = (2a - x) x^2$:
(25) $(x^2 + y^2) y^2 = (ax + by)^2$:
(26) $(x + y)^2 (x^2 + y^2) = 4a^2 x^2$:
(27) $(x^2 \pm y^2)^2 = a^2 x^2 \pm b^2 y^2$:
(28) $a^3 x^3 + b^3 y^3 = a^3 b^3$:
(29) $(x + a) y^2 = x^3 - a^3$:
(30) $y^2 (x - a) = x^3 - b^3$:
(31) $a^2 xy = x^4 - b^4$:
(32) $ay^2 (x - a) = x^4$:
(33) $ax = b (y^2 + y)$:
(34) $a^2 y = x^2 (b - x)$:
(35) $a^2 y = x (b^2 - x^2)$:
(36) $y (x^2 - b^2) = x^2 (x - a)$.
(37) $(y - x)^4 = x^4 - b^4$:
(38) $(y^2 - 1)^2 = (2x + 3) x^2$.
(39) $axy = x^3 - a^3$:
(40) $x^2 y^2 = (x^2 - y^2)^2$:
(41) $(x^2 - a^2) y = bx^2$:
(42) $x^2 y = a^2 x - b^3$:

(43) $(1-x^2)\, y^2 = x^2\, (1-e^2x^2)$:
(44) $(y^2 - b^2)^3 = a^3\, (x^3 - b^3)$:
(45) $xy^2 = (x-9)\, (x-2)^2$:
(46) $(a^2 + x^2)\, y = x^3$:
(47) $a^2\, (x^2 - y^2)^2 = (x^2 + y^2)^3$:
(48) $(x+1)^2\, y = (x-1)\, x^2$:
(49) $ay^2 = x\, (b^2 + x^2)$:
(50) $ay^3 - x^3y = bx^3$:
(51) $(x^2 + y^2)\, y = a\, (x+y)^2$:
(52) $ay^3 + bx^3 = abxy$:
(53) $(y-x)^2 = a^2 - x^2$:
(54) $(x-a)\, y^2 = (x+a)\, x^2$:
(55) $(x+a)\, y^2 = b^2\, (x + 2a)$:
(56) $(a-x)\, y^2 = (a-2x)^2\, x$:
(57) $x^4 + y^4 = 2ay^3 - 2bx^2y$:
(58) $(xy+1)^2 = (2-x)\, (x-1)^3$:
(59) $y = \sin x$:
(60) $y = \sin x \pm \cos x$:
(61) $y = \sin x + 2 \sin 2x$:
(62) $y = \sec x - \tan x$:
(63) $y = \cos^{-1} (1-x)$:
(64) $y = \tan^{-1} (a+x)$:
(65) $ay = \sin^2 x$:
(66) $y = (\log x)^3$:
(67) $y = e^{\cos x}$:
(68) $y = e^{\tan^{-1} x}$:
(69) $y = bx + (x-a)^{\frac{5}{2}}$:
(70) $y = b + cx^2 + (x-a)^{\frac{5}{2}}$:
(71) $(a+x)^2 y^2 = x^2 \left\{b^2 - (a+x)^2\right\}$:
(72) $(2a+x)\, y^2 = (2a-x)\, (a+x)^2$.

63. Investigate the figure of the curve belonging to the equation $y^4 - 2a^2y^2 - 2ax^3 - 3a^2x^2 + a^4 = 0$: and find the positions and natures of all its singular points.

64. Trace the curve defined by the equation

$$(y^2 - ax)^2 = x^2\, (4a^2 - x^2):$$

and find the positions and natures of all its singular points.

65. Determine the figure of the curve whose equation is $\left(\frac{y}{x}\right)^2 = \frac{b^2 - 2ax + x^2}{2ax - x^2}$; and explain the natures of its singular points.

66. Trace the curve and find its singular points when the equation is $(y^2 - a^2)^2 = ax^2\, (2x + 3a)$.

67. Trace the curve whose equation is $a^m y = x\, (x-a)^m$; both when m is an even and an odd number: and find the numbers and natures of its singular points.

68. Investigate the form of the curve whose equation is $y = 2\sin 2x - \sin x$: and find after what values of x the form of the figure recurs.

69. Let it be required to trace the curve whose equation is $x^4 - ax^2y + by^3 = 0$, distinguishing particularly its singular points.

70. Trace the curve belonging to the equation

$$a^3(y-a)^2 = (x-b)^4(x-c),$$

explaining fully the natures of its singular points.

71. If the equation of a curve be

$$a^5(y-a)^2 = (x-b)^6(x-c),$$

it is required to trace it, and to distinguish fully the natures of of its singular points.

72. Trace the curve belonging to the equation

$$a^3y^2 - 2abx^2y = x^5;$$

and particularize the nature of it at the origin of co-ordinates.

73. Investigate the figure of the curve belonging to the equation $a^4 + x^4 + y^4 = 2a^2(x^2 + y^2)$: and find the natures of its multiple points.

74. Trace the curve whose equation is

$$(y^2 + x^2)^2 - 6axy^2 = ax^2(2x - a):$$

and find the natures of its singular points.

75. Trace the curve whose equation is

$$xy^2 = a(x-b)(x-c);$$

and distinguish the different cases depending upon the relative values of b and c, regarded as possible and imaginary.

76. If the equation of a curve be $xy^2 = (a-x)(x-b)(x-c)$, it is required to trace it, and distinguish the various forms of which it admits.

77. Investigate the figure of the curve called the *Trident*, as defined by the equation $xy = ax^3 + bx^2 + cx + e$: and distinguish the different cases originating from the forms of the roots of the equation $ax^3 + bx^2 + cx + e = 0$.

78. Investigate the form of the curve expressed generally by the equation $xy^2 + ay = b + cx$: and distinguish particularly the cases wherein c is positive, negative and evanescent.

Sir Isaac Newton in his *Enumeratio Linearum tertii Ordinis*, designates the curve belonging to the equation last given, by the name of the *Hyperbolism* of an *Hyperbola, Ellipse* or *Parabola* according as c is positive, negative or evanescent; and the reason he assigns for it is the following:

Solving the equation proposed with respect to y, we have

$$y = \frac{-a \pm \sqrt{a^2 + 4bx + 4cx^2}}{2x}:$$

and if the denominator of this expression were an invariable quantity, it is obvious that the equation would belong to an hyperbola, ellipse or parabola, according as c is positive, negative or evanescent: whence, if such constant quantity be replaced by the variable quantity $2x$, the conic section becomes *hyperbolized* by having in each case an infinite branch at the origin of its co-ordinates.

79. Trace the curve defined by the polar equation $r \cos^2 \theta = a \sin \theta$. See the second Diagram of page (304).

80. Find the figure of the polar curve defined by the equation $r^2 \cos^3 \theta = a^2 \sin \theta$. See the Diagram of page (303).

81. Trace the rectangular hyperbola as defined by the equation $r^2 \sin 2\theta = 2a^2$. See the first Diagram of page (289).

82. Investigate the figure of the curve whose equation is $r^2 \cos^4 \theta = a^2 \cos 2\theta$. See the Diagram of page (294).

83. Find the figure of the curve defined by the equation $r = a \sec \theta - b$. See the Diagram of page (312).

84. Describe the figure of the curve whose equation is $r^2 = a^2 \sin 2\theta$. See the Diagram of (294), with the direction of the fixed axis changed through half a right angle.

85. Trace the figure of the reciprocal spiral whose equation is $r = a\theta^{-1}$.

86. Determine the figure of the logarithmic spiral whose equation is $y = a^{\theta}$.

87. Describe the lituus as defined by the equation $r^2\theta = a^2$.

88. Trace the circle as defined by the equation

$$r = a(\cos\theta - \sin\theta):$$

and find the values of its *maximum* and *minimum* radius vector.

89. Describe the figure of the rectangular hyperbola as expressed by the equation $r = a(\operatorname{cosec}\theta - \sec\theta)$.

90. Trace out the figure of the cissoid of *Diocles* by means of the equation $r\cos\theta = 2a\sin^2\theta$.

91. Find the form of the cardioide by means of its equation $r = a(1 + \cos\theta)$.

92. Required the form of the figure called the trisectrix, whose equation is $r = a(2\cos\theta \pm 1)$.

93. It is required to trace out the figure of a rectangular hyperbola by means of the equation $r\cos 2\theta = 2a\cos\theta$.

94. Required to trace out the figure belonging to the equation $r\sin(\alpha + \theta) = a\sin\alpha$.

95. It is required to give graphical representations of the curves defined by the following equations:

(1) $r = a\theta$:
(2) $r = a\sin 2\theta$:
(3) $r^2 = a^2\tan 2\theta$:
(4) $r = a(\sec\theta - \tan\theta)$:
(5) $r = a\cos\theta$:
(6) $r = a\sec\theta\tan^2\theta$:
(7) $r = a\tan\theta$:
(8) $r = a\sec\theta$:
(9) $r^2 = a^2(\tan^2\theta - 1)$:
(10) $r = a(1 + 2\sin\frac{1}{2}\theta)$:
(11) $r = a\log\tan(45 + \frac{1}{2}\theta)$:
(12) $r(e^{\theta} + e^{-\theta}) = 2a$:
(13) $r^2(e^{\theta} - 1)^2 = 4a^2e^{\theta}$:
(14) $r(\theta^2 - 1) = a\theta^2$:
(15) $r\sin 2\theta = a\sin\theta\sin 3\theta$:
(16) $r(\cos\theta + \cot\theta) = 2a$.

96. Trace the curve whose equation is $y^3 - 3axy + x^3 = 0$, by assuming $y = vx$, and therefore $x = \dfrac{3av}{v^3 + 1}$.

97. By means of the same assumption, determine the form of the curve whose equation is $y^5 = ax^4 + mx^5$.

98. After the same manner it is required to trace the curve defined by the equation $x^5 + a^3xy - y^5 = 0$.

99. Apply the same principle to the describing of the curve whose equation is $ax^2 + xy^2 + x^3 - y^3 = 0$.

CHAP. XII.

MISCELLANEOUS EXAMPLES.

By reference to the Principles laid down and exemplified in Articles (256)—(282), the following Theorems and Problems may be solved.

1. Given $u = x^2 (a+y)^3$, to find the corresponding value of $u' = (x+h)^2 (a+y+k)^3$, by means of the Theorem of (259).

2. Apply the same Theorem to find the values of u' when $u = x^3 \log y$ and $u = x^2 \sin 2y$.

3. Also, to find the values of u' from the equations $u = x^m (\log y)^n$ and $u = e^x (\sin y)^m$.

4. Again, to determine the values of u' if $u = \sin x e^{\cos y}$, $u = x^m \log \sin e^y$ and $u = e^{xy} \sin (x+y)$.

5. It is required to verify the formulæ established in Article (261), in the following instances:

(1) $u = \dfrac{x}{x+y}$: (2) $u = \dfrac{x-y}{x+y}$:

(3) $u = \dfrac{x^2}{x^2-y^2}$: (4) $u = \dfrac{x^2-y^2}{x^2+y^2}$:

(5) $u = \dfrac{x^2+y^2}{x+y}$: (6) $u = \dfrac{2xy}{x^3-y^3}$:

(7) $u = e^{xy}(e^x + e^y)$: (8) $u = \log \left(\dfrac{x+y}{x-y}\right)$:

(9) $u = e^{x^y}$: (10) $u = \sin^{-1} \sqrt{\dfrac{x-y}{x}}$:

(11) $u = \log \dfrac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}}$: (12) $u = \log \tan \left(\dfrac{x}{y}\right)$:

(13) $u = x \sin y - y \sin x$: (14) $u = \dfrac{y}{x} \log \dfrac{x}{y}$:

(15) $u = (\sin x)^{\cos y}$: (16) $u = \cos \{\cos x + \cos y\}$.

6. If $u = \dfrac{x^m}{y^n}$, it is required to prove that

$$du = \frac{x^{m-1}}{y^{n+1}} (my\,dx - nx\,dy).$$

7. If $u = \dfrac{ay}{\sqrt{x^2 + y^2}}$, then will $du = -\dfrac{axy\,dx - ax^2 dy}{(x^2 + y^2)^{\frac{3}{2}}}$.

8. If $u = \sqrt{\dfrac{x^2 - y^2}{x^2 + y^2}}$, then $du = \dfrac{2xy\,(y\,dx - x\,dy)}{(x^2 + y^2)\sqrt{x^4 - y^4}}$.

9. If $u = \log \dfrac{x + \sqrt{x^2 - y^2}}{x - \sqrt{x^2 - y^2}}$, then $du = \dfrac{2\,(y\,dx - x\,dy)}{y\sqrt{x^2 - y^2}}$.

10. If $u = \tan^{-1} \dfrac{x}{y}$, then will $du = \dfrac{y\,dx - x\,dy}{x^2 + y^2}$.

11. If $u = \sin^{-1} \sqrt{\dfrac{x - y}{x + y}}$, $du = \dfrac{y\,dx - x\,dy}{(x + y)\sqrt{2y\,(x - y)}}$.

12. If $u = \tan \dfrac{x}{y}$, then will $du = \dfrac{y\,dx - x\,dy}{y^2} \sec^2 \dfrac{x}{y}$.

13. If $u = \sin^{-1} \dfrac{x}{y}$, then will $du = \dfrac{y\,dx - x\,dy}{y\sqrt{y^2 - x^2}}$.

14. If $u = x^y$, then will $du = yx^{y-1}dx + x^y \log x\,dy$.

15. If $u = x \sin y + y \sin x$, then we shall have

$$du = (\sin y + y \cos x)\,dx + (\sin x + x \cos y)\,dy.$$

16. Let it be required to find the total differentials of the following functions of two independent variables:

(1) $u=x^3-3axy+y^3$: (2) $u=2y^4+3xy^3+x^2y^2$:

(3) $u=x+y+\sqrt{2xy+y^2}$: (4) $u=x^2+y^2+a\sqrt{x^2-y^2}$:

(5) $u=\dfrac{x}{x+y}$: (6) $u=\dfrac{x^3+y^3}{x-y}$:

(7) $u=\dfrac{x-y}{\sqrt{x^2+y^2}}$: (8) $u=\dfrac{xy+\sqrt{x^2+y^2}}{xy-\sqrt{x^2+y^2}}$:

(9) $u=\log\,(x+\sqrt{x^2+y^2})$: (10) $u=\log\,(xy-\sqrt{x^2y^2-1})$:

(11) $u=\sin\dfrac{x-y}{x+y}$: (12) $u=\tan^{-1}\dfrac{x+y}{x-y}$:

(13) $u=\cot x^y$: (14) $u=\sin^{-1}\dfrac{xy}{\sqrt{x^2+y^2}}$:

(15) $u=\dfrac{y}{x-y}-\log\left(\dfrac{y-x}{c}\right)$: (16) $u=\sec\dfrac{x}{y}\sec^{-1}\dfrac{y}{x}$:

(17) $u=\sqrt{x^2+y^2}+a\tan^{-1}\dfrac{x}{y}$: (18) $u=\sin^{-1}(\cos x-\sin y)$.

17. If $u=xyz$, then it is required to prove that

$$du=yzdx+xzdy+xydz.$$

18. If $u=xy-xz+yz$, it is required to shew that $du=(y-z)\,dx+(x+z)\,dy-(x-y)\,dz$.

19. If $u=\dfrac{xz-z^2}{y^2}$, it is required to prove that

$$du=\frac{1}{y^3}(yzdx+2z^2dy-2xzdy+xydz-2yzdz).$$

20. If $u=\dfrac{xy}{ax+bz}$, $du=\dfrac{ax^2dy+bxzdy+byzdx-bxydz}{(ax+bz)^2}$.

21. If $u=\sqrt{x^2-y^2+z^2}$, then $du=\dfrac{xdx-ydy+zdz}{\sqrt{x^2-y^2+z^2}}$.

22. If $u=a\phi\,(x)+by$, then $du=a\phi'\,(x)\,dx+bdy$

23. If $u=x^2\phi\,(y)+cx$, then will

$$du=\{2x\phi\,(y)+c\}\,dx+x^2\phi'\,(y)\,dy.$$

24. If $u = x(x+z) + y(y+z)$, then will

$$du = (2x+z)\,dx + (2y+z)\,dy + (x+y)\,dz.$$

25. Required the total differentials of the following functions of more than two independent variables:

(1) $u = \dfrac{x}{y+z}$: (2) $u = \dfrac{xyz}{x^2 - z^2}$:

(3) $u = \dfrac{(x+z)\,y}{y+z}$: (4) $u = \dfrac{xyz}{x-y+z}$:

(5) $u = \log\,(x+y)^2\,(y+z)$: (6) $u = \sin\left(\dfrac{x^2+y^2}{x^2+z^2}\right)$:

(7) $u = \dfrac{x^2y^2}{z^2+t^2}$: (8) $u = \dfrac{txyz}{t+x+y+z}$:

(9) $u = \phi\left(xy,\, \dfrac{y}{t}\right)$: (10) $u = f\left(\dfrac{x}{y},\, \dfrac{y}{z},\, \dfrac{z}{t}\right)$.

26. If $u = x^3 + x^2y + xy^2 + y^3$, it is required to prove in accordance with Article (272) that

$$3u = \frac{du}{dx}\,x + \frac{du}{dy}\,y;$$

$$6u = \frac{d^2u}{dx^2}\,x^2 + 2\,\frac{d^2u}{dx\,dy}\,xy + \frac{d^2u}{dy^2}\,y^2;$$

&c.....................................

27. It is required to verify the same formulæ in the following instances:

(1) $u = x^2 + y^2$: (2) $u = (x+y)(x^3 - y^3)$:

(3) $u = \dfrac{x^3 - y^3}{x+y}$: (4) $u = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$:

(5) $u = x\sqrt{2xy + y^2}$: (6) $u = x - y + \sqrt{x^2+y^2}$:

(7) $u = \sin\sqrt{\dfrac{x+y}{x-y}}$: (8) $u = \tan^{-1}\sqrt[3]{\dfrac{x^2-y^2}{x^2+y^2}}$:

(9) $u = \log\sec^{-1}\sqrt[m]{\dfrac{x^m+y^m}{x^m-y^m}}$: (10) $u = \dfrac{x+y}{\sin\,(x-y)}$:

(11) $e^u = x + y + \sqrt{2xy + y^2}$: (12) $u = e^{xy}\tan^{-1}(x+y)$.

28. If $(x+y+z)\,e^{z}=c$, prove that by eliminating the constant quantity and exponential function we shall have

$$dx+dy+dz+(x+y+z)\,dz=0.$$

29. If $xz+yt=c\,(y+z)$, it is required to prove that $(yz+z^2)\,dx+(zt-xz)\,dy+(xy-yt)\,dz+(y^2+yz)\,dt=0$.

30. If $u=f(z)$ and $z=\phi\,(x,\,y)$, it is required to prove that $\frac{du}{dx}:\frac{du}{dy}=\frac{dz}{dx}:\frac{dz}{dy}$; and thence to shew that the indeterminate function f of a determinate function z of two variables x and y may be eliminated.

31. If the indeterminate function be eliminated from the equation $z=\frac{x}{a}+f(ay-bx)$, it is required to prove that

$$a\frac{dz}{dx}+b\frac{dz}{dy}=1.$$

32. If $z=\phi\left(\frac{x}{y}\right)$, then will $px+qy=0$, where

$$p=\frac{dz}{dx} \text{ and } q=\frac{dz}{dy}.$$

33. If $z=a\sqrt{x^2+y^2}+\phi\left(\frac{x}{y}\right)$, then $px+qy=a\sqrt{x^2+y^2}$.

34. If $z=xy+\phi\left(\frac{x}{y}\right)$, then will $pxz+qyz=xy$.

35. If $z=\phi\left(\frac{y}{a}+\log x\right)$, then will $ap-qx=0$.

36. If $z=\frac{a}{m}\log y+\phi\left(\frac{x^m}{y^n}\right)$, then will $npx+mqy=a$.

37. If $z=(x+y)^m\,\phi\,(x^2-y^2)$, then will $py+qx=mz$.

38. If $z=\frac{y^2}{3x}+\phi\,(xy)$, then will $qxy=px^2+y^2$.

39. If $zx = m\phi\left(\frac{y}{x}\right)$, then will $px + qy + z = 0$.

40. If $z = \phi\left(\frac{y^2 - x^2}{x}\right)$, then will $2pxy + q(x^2 + y^2) = 0$.

41. If $y^2 - z^2 = 2\phi\left(ay + \frac{x^2}{2}\right)$, then $apz - qxz + xy = 0$.

42. If $x^2 + y^2 + z^2 = \phi(ax + by + z)$, then will

$$(y - bz)p - (x - az)q = bx - ay.$$

43. If $x^{m-1}(x + y + z) = \phi\left(\frac{x}{y}\right)$, then will

$$m(x + y + z) = z - px - qy.$$

44. If $z = e^{\frac{y}{a}} f(x - y)$, then will $a(p + q) = z$.

45. If $z^2 = x^2 + \phi\left(\frac{x}{y}\right)$, then will $pxz + qyz = x^2$.

46. If $z = e^{\frac{x}{x+y}} f(x + y)$, then will $(p - q)(x + y) = z$.

47. If $\sin^{-1}\frac{z}{\sqrt{y^2 + z^2}} = \log\left\{\frac{1}{x}\phi(y^2 + z^2)\right\}$, then will

$$px + qz + y = 0.$$

48. If $z = t^m f\left(\frac{x}{t}, \frac{y}{t}\right)$, then we shall have

$$px + qy + vt = mz, \text{ if } v = \frac{dz}{dt}.$$

49. If $(m - 1)z + \frac{xy}{t} = x^m \phi\left(\frac{y}{x}, \frac{t}{x}\right)$, then will

$$px + qy + vt = mz + \frac{xy}{t}.$$

50. If $z = \phi\left\{(x - y)^2 (x + y + t),\ (x - t)^2 (x + y + t)\right\}$, then will $p(t + y) + q(t + x) + v(x + y) = 0$.

51. If $z + y + t = y^2\phi\left\{(xz - xy),\ (xy - xt)\right\}$, then will $px + q(z + t) + v(z + y) = t + y$.

52. Required to eliminate the indeterminate function $\phi\left(\frac{x^2-y^2}{x^2+y^2}\right)$ from the equation $\frac{zy}{x}=\phi\left(\frac{x^2-y^2}{x^2+y^2}\right)$.

53. Eliminate the indeterminate function $f\left(\frac{x^2}{x^2+y^2}\right)$ from the equation $z\phi(xy)=f\left(\frac{x^2}{x^2+y^2}\right)$: and shew what the result becomes when $\phi(xy)=\sin xy$.

54. Eliminate the indeterminate function from the equation $\frac{z-\gamma}{x-a}=f\left(\frac{x-a}{y-\beta}\right)$: and find what the result becomes when $a=\beta=\gamma=0$.

55. If $z=\frac{ax}{2y}-\frac{f(y)}{x^{m-1}}+\phi(y)$, it is required to prove that $rxy+2py=a$, where $r=\frac{d^2z}{dx^2}$.

56. If $z=\frac{1}{6}x^3y^2+f(x)+\phi(y)$, it is required to shew that $s-x^2y=0$, where $\frac{d^2z}{dx\,dy}=s$.

57. If $z=\frac{ay\log x}{m+1}+\frac{f(x)}{y^m}+\phi(y)$, it is required to prove that $sxy+mpx-ay=0$.

58. If $z=\frac{4}{15}(x+y+z)^{\frac{5}{2}}+f(y,z)+\phi(x,z)$, it is required to prove that $s^2=x+y+z$.

59. If $z=\frac{1}{8}x^2y^2t^2+F(x,y)+f(x,t)+\phi(y,t)$, it is required to prove that $\frac{d^3z}{dx\,dy\,dt}=xyt$.

60. If $z=F(y+ax)+xf(y+ax)+\phi(y+bx)$, then

$$\frac{d^3z}{dx^3}-(2a+b)\frac{d^3z}{dx^2\,dy}+(a^2+2ab)\frac{d^3z}{dx\,dy^2}-a^2b\frac{d^3z}{dy^3}=0.$$

61. Let it be required to eliminate both the indeterminate functions from the equation $z=xf\left(\frac{y}{x}\right)+\phi\left(\frac{y}{x}\right)$.

62. Eliminate the indeterminate functions from the equation $z = x^m F\left(\frac{y}{x}\right) + y^m f\left(\frac{y}{x}\right)$.

63. From the equation $z = F(y + ax) f(y - ax)$, it is required to eliminate the indeterminate functions F and f.

64. Eliminate the indeterminate, exponential, &c. functions from the following equations:

(1) $zx = xy + f\left(\frac{y}{x}\right)$: (2) $z = \frac{cy}{a} + f(ax - by)$:

(3) $z = \frac{1}{2}x^2 + f(y + \log x)$: (4) $z = y \sin^{-1}\frac{x}{y} + \phi(y)$:

(5) $z = f(y) - \sqrt{y^2 - x^2}$: (6) $\log z = f(xy) - \frac{ax}{3y^2}$:

(7) $(x-y)\, z = e^{\frac{x}{x-y}} f(x-y)^2$: (8) $\sin z = f\left(\frac{y}{x}\right) + \sqrt{1 - \frac{x^2}{y^2}}$:

(9) $z = f(\log x + 2\sqrt{y})$: (10) $z = \log^{-1}\left\{xy - \frac{a^2 + x^2}{y^2}\right\}$:

(11) $z = \sqrt{x^2 + y^2}\, f\left\{\tan^{-1}\frac{x}{y} - \log\sqrt{x^2 + y^2}\right\}$:

(12) $\text{Log}\, z = b(y - ax) + \log\{f(y + ax) + \phi(y - ax)\}$:

(13) $$z = \frac{x(x^2 + 3y^2)}{6y}\log(x^2 + y^2) - \frac{1}{9y}\left(x^3 + 6xy^2 - 6y^3 \tan^{-1}\frac{x}{y}\right)$$
$$+ \tfrac{1}{3}x(x^2 + 3y^2) f(x) + \phi(y):$$

(14) $p + aq = f\left(\frac{y}{x}\right)$: (15) $ap - bq = f(ay + bx)$.

(16) $p = q f(z)$: (17) $p + q = (p - q) f(z)$:

(18) $z = \sqrt{xy}\, f\left(\frac{y}{x}\right) + \phi(xy)$: (19) $z = e^{ax}\{f(y - ax) + \phi y\}$.

65. From the equation $z = cx + f(x + y)$, it is required to prove that $r^2 - t^2 = rt - s^2$.

66. From the equation $1 - y + ay = 0$, it is required to find the values of y^2 and $y^{\frac{1}{2}}$ by *Lagrange's* Theorem.

67. From the same equation determine the values of the functions y^{-m}, $\log y$, $\sin y$, $e^{1-\frac{1}{y}}$, $\log(2y-1)$ and $\sin\left(1-\frac{1}{y}\right)$.

68. It is required to find the sum of the m^{th} powers of the roots of the equation $a - by + cy^2 = 0$.

69. From the equation $1 - y + y^2 = 0$, find the values of the functions y^m, y^{-m}, e^y, $\log y$, $\sin y$, $\sin^{-1} y$ and $\tan\left(\frac{y+1}{y}\right)$.

70. Given the equation $xy^n - y + a = 0$, to find the values of y^2, y^{-2}, $y^{-\frac{1}{2}}$, e^y, $\log y$, $\sin y$, $\cos^{-1} y$ and $\tan^{-1} 3y$.

71. Find all the roots of the equation $y^3 - 9y + 28 = 0$, by means of the Theorem of *Lagrange*.

72. Given the equation $1 - y + a^y = 0$, to find the values of y^m, y^{-m}, $\log y$, $\sin y$ and $\sin^{-1} 2y$.

73. Apply *Lagrange's* Theorem to deduce $\sin y$ from the equation $y = 1 - x \sin y$.

74. Find the values of u, $\sin u$ and $\sin mu$ in terms of θ, from the equation $\theta = u + e \sin u$.

75. Required the value of y from the equation

$$1 + x \sin y = \tan y,$$

by means of *Lagrange's* Theorem.

76. Apply *Lagrange's* Theorem to find the value of z in terms of y, from the equation $ay^3 + zy^3 - az^3 = 0$.

77. From the equation $r = \dfrac{a(1-e^2)}{1 + e \cos\theta}$, it is required to find the values of e and $\log r$ in terms of the rest.

78. Apply *Laplace's* Theorem to find the value of e^y from the equation $y = \log(z + x \sin y)$.

79. If $u^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, it is required to prove that

$$\frac{d^2\left(\frac{1}{u}\right)}{dx^2} + \frac{d^2\left(\frac{1}{u}\right)}{dy^2} + \frac{d^2\left(\frac{1}{u}\right)}{dz^2} = 0.$$

80. If u be a homogeneous function of x, y, z, &c. of m dimensions, it is required to prove that

$$(m-1)\,du = x\,dp + y\,dq + \text{\&c.}$$

where p, q, &c. are the partial differential coefficients of the first order relatively to x, y, &c.

81. Having given $nt = \theta - 2e\sin\theta + \frac{3}{4}e^2\sin 2\theta - \frac{1}{3}e^3\sin 3\theta$, it is required to apply *Lagrange's* Theorem to prove that

$$\theta = nt + (2e - \tfrac{1}{4}e^3)\sin nt + \tfrac{5}{4}e^2\sin 2nt + \tfrac{13}{12}e^3\sin 3nt,$$

where the powers of e higher than e^3 are neglected.

CHAP. XIII.

By means of the Principles laid down and illustrated in Articles (283)—(289), it is required to solve the following Theorems and Problems.

1. If $u = x^2y + xy^2 - axy$, it is required to prove that when $x = \frac{1}{2}a$ and $y = \frac{1}{2}a$, the value of $u = 0$, which is a *minimum*.

2. If $u = x^3 - 3axy + y^3$, then will $x = a$ and $y = a$ render $u = -a^3$, a *minimum:* also, if $x = 0$ and $y = 0$, then $u = 0$, which is neither a *maximum* nor a *minimum:* required a proof.

3. It is required to prove that the function

$$u = a^2bx + a^2cy - bx^2y - cxy^2,$$

admits of neither a *maximum* nor a *minimum*.

4. In the equation $u = x^3y^3 + x^2y^4 - ax^2y^3$, it is required to shew that $u = -\dfrac{a^6}{432}$, is a *minimum* when $x = \frac{1}{3}a$ and $y = \frac{1}{2}a$.

5. If $u = a^2x^2y - 2ax^3y + x^4y - 2ax^2y^2 + 2x^3y^2 + x^2y^3$, it is required to prove that corresponding to $x = \frac{2}{5}a$ and $y = \frac{1}{5}a$, the value of u is a *maximum*, and find it.

6. If $u = x^3y + xy^3 - a^2x^2 - a^2y^2$, it is required to shew that neither a *maximum* nor a *minimum* can take place.

7. If $u = x^2y + xy^2 - ax^2 - ay^2 - 3axy + 2a^2x + 2a^2y$, it is required to prove that corresponding to $x = \frac{2}{3}a$ and $y = \frac{2}{3}a$, the value of u is a *maximum*, and find it.

8. If $u=2a^2xy-3ax^2y-ay^3+x^3y+xy^3$, it is required to prove that when $x=\frac{3}{2}a$ and $y=\frac{1}{2}a$, the value of $u=-\frac{7}{8}a^4$, a *minimum:* when $x=\frac{3}{2}a$ and $y=-\frac{1}{2}a$, the value of $u=\frac{7}{8}a^4$, a *maximum:* when $x=\frac{1}{2}a$ and $y=\frac{1}{2}a$, the value of $u=\frac{1}{8}a^4$, a *maximum:* and when $x=\frac{1}{2}a$ and $y=-\frac{1}{2}a$, the value of $u=-\frac{1}{8}a^4$, a *minimum.*

9. Given $u=x^4+y^4-4axy^2$, it is required to shew that when $x=a\sqrt{2}$ and $y=2^{\frac{3}{4}}a$, the value of $u=-4a^4$, a *minimum.*

10. If $u=xy^2(a-x-y)^3$, it is required to prove that when $x=\frac{1}{3}a$ and $y=\frac{2}{3}a$, the value of $u=\frac{4}{27}a^6$, a *maximum.*

11. If $u=x^2y(x^4+2y^3-a)^2$, it is required to prove that corresponding to $x=\sqrt[4]{\frac{3a}{17}}$ and $y=\sqrt[3]{\frac{a}{17}}$, the value of u is a *maximum.*

12. Given $u=y^4-8y^3+18y^2-8y+x^3-3x^2-3x$, it is required to shew that when $x=1-\sqrt{2}$ and $y=2$, the value of $u=3+4\sqrt{2}$, a *maximum;* and when $x=1+\sqrt{2}$ and $y=2\pm\sqrt{3}$, the value of $u=-6-4\sqrt{2}$, a *minimum.*

13. Given $u=y^4-8y^3+18y^2-8y-x^3+3x^2+3x$, it is required to shew that when $x=1+\sqrt{2}$ and $y=2$, the value of $u=13+4\sqrt{2}$, a *maximum:* and when $x=1-\sqrt{2}$ and $y=2\pm\sqrt{2}$, the value of $u=4-4\sqrt{2}$, a *minimum.*

14. If $u=xy+\frac{a^3}{x}+\frac{a^3}{y}$, then if $x=a$ and $y=a$, it is required to prove that $u=3a^2$ is a *minimum.*

15. If $u=\sin x+\sin y+\sin(x+y)$, it is required to prove that when $x=60^0$ and $y=60^0$, the value of $a=\frac{3}{2}\sqrt{3}$, a *maximum.*

16. If $u=\sin x \sin y \sin(x+y)$, it is required to shew that when $x=60^0$ and $y=60^0$, $u=\frac{3}{8}\sqrt{3}$, a *maximum.*

17. Find whether the function $u=x^2y^3(a-2x-3y)^4$ admits of a *maximum* or a *minimum*, and determine its value.

18. Determine whether the function $u=x^3+ay^2-bxy+cx$, admits of a *maximum* or a *minimum*, and find its value.

19. Given $u=\sqrt{(x-a)^2+y^2+b^2}+\sqrt{(x-a)^2+y^2-\beta^2}$, to find whether u admits of a *maximum* or a *minimum*, and when possible, to determine its value.

20. Determine the different *maximum* or *minimum* values of the function $u=x^2+y^2+b+\sqrt{a^2-(x-c)^2-y^2}$, and apply the criterion to decide which they are.

21. Find what *maximum* or *minimum* values the function $u=x^4+y^4-ax^2y-axy^2+c^2x^2+c^2y^2$ admits of, and decide which they are.

22. In the equation $u=a\sin x+b\sin y+c\sin(x+y)$, it is required to deduce the equations by means of which a *maximum* or *minimum* value of u may be ascertained.

23. If $u=xy\sqrt{a^2b^2-a^2x^2-b^2y^2}$, it is required to find its *maximum* or *minimum* value, and to apply the criterion for deciding which it is.

24. If $u=axy^2z^3-x^2y^2z^3-xy^3z^3-xy^2z^4$, it is required to prove that when $x=\frac{1}{7}a$, $y=\frac{2}{7}a$ and $z=\frac{3}{7}a$, the value of $u=\left(\frac{a}{7}\right)^7$, a *maximum*.

25. Given $u=ax^2y^3z^4-x^3y^3z^4-x^2y^4z^4-x^2y^3z^5$, it is required to prove that when $x=\frac{1}{5}a$, $y=\frac{3}{10}a$ and $z=\frac{2}{5}a$, the value of $u=$ a *maximum*.

26. It is required to prove that the function

$$u=ax^2-bxy+cxz+yz,$$

admits of neither a *maximum* nor a *minimum*.

27. Required the values of x, y and z, which shall render the function $u=\dfrac{xyz}{(a+x)(x+y)(y+z)(z+b)}$, a *maximum* or a *minimum*.

28. It is required to find the *maximum* or *minimum* values of $u = \left(\frac{a}{x}\right)^m + \left(\frac{b}{y}\right)^m$, subject to the condition expressed by the equation $x^n + y^n = c^n$.

29. Find the *maximum* or *minimum* value of

$$u = m \cos^2 x + n \cos^2 y,$$

when the equation of condition is $x + y = a$.

30. Required the *maximum* and *minimum* values of

$$u = \sin^m x \cos^n y,$$

when x and y are connected by the equation $x - y = a$.

31. Required the *maximum* or *minimum* value of

$$u = (mx + n)\,(ny + m),$$

subject to the equation of condition $a^{mx} b^{ny} = c$.

32. Find the *maximum* value of $u = xyz$, subject to the condition expressed by the equation $x + y + z = k$.

33. It is required to determine the *maximum* and *minimum* values of u in the equation

$$u = \left(\frac{a}{x}\right)^m + \left(\frac{b}{y}\right)^m + \left(\frac{c}{z}\right)^m,$$

under the condition that $x^n + y^n + z^n = k^n$.

34. Determine the *maximum* value of $u = xy^2z^3$, when $x + my^2 + nz^3 = a$, is the equation of condition.

35. It is required to find the *maximum* value of $u = x^4yz^2$, when x, y and z are connected by the equation of condition $x^2 + 2y^3 + z^4 = a$.

36. Required the *maximum* or *minimum* value of

$$u = ax^\alpha + by^\beta + cz^\gamma,$$

when $x^{\alpha'} + y^{\beta'} + z^{\gamma'} = k$ is the equation of condition.

37. Find the *maximum* or *minimum* value of $u = xyz$, subject to the condition expressed by the equation $a^x b^y c^z = k$.

38. Required the *maximum* or *minimum* value of

$$u = (x + a)\,(y + b)\,(z + c),$$

the equation of condition being $a^x b^y c^z = k$.

39. Determine the *maximum* or *minimum* value of

$$u = (a^x - 1)\,(b^y - 1)\,(c^z - 1),$$

when the equation of condition is $a^x b^y c^z = k$.

40. It is required to find the *maximum* or *minimum* value of $u = \left(\frac{x-1}{x}\right)\left(\frac{y-1}{y}\right)\left(\frac{z-1}{z}\right)$, when the equation of condition is $x^\alpha y^\beta z^\gamma = k$.

41. What is the *maximum* value of the function

$$u = x^\alpha y^\beta z^\gamma t^\delta,$$

when the quantities x, y, z and t are subject to the condition expressed by the equation $x + 2y + 3z + 4t = a$?

42. If $x + y + z = \frac{1}{2}\pi$, it is required to find the value of $u = \sin x \sin^2 y \sin^3 z$, when it is a *maximum* or a *minimum*.

43. Find the *maximum* or *minimum* value of

$$u = \sin x \sin y + \sin x \sin z + \sin y \sin z,$$

when the equation of condition is $x + y + z = 45^0$.

44. Determine whether the function $x^m \log y \sin z$, admits of a *maximum* or a *minimum*, when the equation of condition is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

45. In the equation $u = a - (x^2 + y^2)^{\frac{3}{2}}$, it is required to prove by the substitution of $0 \pm h$ and $0 \pm k$ in the places of x and y, that when $x = 0$ and $y = 0$, the value of $u = a$ is a *maximum*.

CHAP. XIV.

MISCELLANEOUS EXAMPLES.

By means of the Principles laid down and exemplified in Articles (290)—(343), the solutions of the following Theorems and Problems may be obtained.

1. If the equation of a sphere be $x^2 + y^2 + z^2 = a^2$, then will the equation to its tangent plane be $xx' + yy' + zz' = a^2$: required a proof.

2. Determine the inclinations of the tangent plane last found to each of the co-ordinate planes.

3. Let it be required to construct the plane touching a sphere by means of the equation above found.

4. To a given sphere it is required to draw a tangent plane which shall pass through a given point, and to shew how far the position of such plane is indeterminate.

5. Required the equation to a plane passing through two given points and touching a given sphere.

6. Draw a tangent plane to a spheroid defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$: shew how it may be constructed and determine its inclinations to the co-ordinate planes.

7. To the surface last mentioned it is required to draw tangent planes which shall pass through one and two given points.

8. A conical surface whose vertex is the origin of co-ordinates being defined by the equation $z = e\sqrt{x^2 + y^2}$, it is required to prove that its tangent plane will be defined by the equation $zz' = e^2 (xx' + yy')$.

9. Construct the tangent plane thus defined, and prove that it always passes through the origin of co-ordinates.

10. It is required to prove that the tangent plane last found always cuts the axis of z at an angle whose tangent is e: find also the angles at which it meets the co-ordinate planes.

11. The equation of an ellipsoid being $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, it is required to prove that the sines of the inclinations of its tangent plane to the co-ordinate planes are

$$\frac{x}{a^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}},\quad \frac{y}{b^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}} \text{ and } \frac{z}{c^2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}.$$

12. In the same surface, it is required to shew that the length of the perpendicular upon the tangent plane from the origin of co-ordinates is $\frac{1}{\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}$.

13. In the same surface it is required to shew that the area of the tangent plane intercepted between the co-ordinate planes is represented by $\frac{a^2b^2c^2}{2xyz}\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$.

14. In the same surface, it is required to prove that the volume of the pyramid formed by the tangent plane and the three co-ordinate planes is represented by $\frac{1}{6}\frac{a^2b^2c^2}{xyz}$.

15. The equation of the elliptic paraboloid being

$$z=\frac{x^2}{a}+\frac{y^2}{b},$$

it is required to prove that the equation to its tangent plane is

$$\frac{2xx'}{a}+\frac{2yy'}{b}-z'=z.$$

16. Let it be required to construct the plane last defined: to find its inclinations to the co-ordinate planes: the perpendicular upon it from the origin: the area of the tangent plane, and the volume of the solid included between it and the co-ordinate planes.

17. The equation of the hyperbolic paraboloid being $z = \frac{x^2}{a} - \frac{y^2}{b}$, it is required to prove that the equation to its tangent plane is $\frac{2xx'}{a} - \frac{2yy'}{b} - z' = z$.

18. Construct the plane last determined: find its inclinations to the co-ordinate planes: the perpendicular upon it from the origin: the area of the tangent plane, and the volume of the solid comprised between it and the co-ordinate planes.

19. The hyperboloid of one sheet being defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, it is required to prove that the equation to its tangent plane is $\frac{xx'}{a^2} + \frac{yy'}{b^2} - \frac{zz'}{c^2} = 1$.

20. It is required to construct the plane last defined; to find its inclinations to the co-ordinate planes: the length of the perpendicular upon it from the origin: the area of the tangent plane, and the volume included between it and the co-ordinate planes.

21. Of the hyperboloid of two sheets defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1,$$

it is required to prove that the equation to the tangent plane is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} - \frac{zz'}{c^2} = -1.$$

22. Construct the plane just determined: find its inclinations to the co-ordinate planes: the length of the perpendicular let fall upon it from the origin: the area of the tangent plane, and the volume of the solid included between it and the co-ordinate planes.

23. The cono-cuneus of *Wallis* being defined by the equation $\frac{x}{\sqrt{a^2-z^2}}+\frac{y}{b}=1$, it is required to prove that the equation to a plane touching it is

$$b(a^2-z^2)x'+(a^2-z^2)^{\frac{3}{2}}y'+bxzz'=a^2bx+(a^2-z^2)^{\frac{3}{2}}y.$$

24. In the groin whose equation is $\frac{x}{a}+\frac{y}{b}=\frac{\sqrt{a^2-z^2}}{a}$, it is required to prove that the tangent plane will be defined by the equation $\frac{x'}{a}+\frac{y'}{b}+\frac{zz'}{a\sqrt{a^2-z^2}}=\frac{a}{\sqrt{a^2-z^2}}$.

25. It is required to construct this plane: to find its inclinations to the co-ordinate planes: the length of the perpendicular let fall upon it from the origin: the area of the tangent plane, and the volume of the solid comprised between it and the co-ordinate planes.

26. If a surface be defined by the equation $xyz=a^3$, then will the equation to its tangent plane be $\frac{x'}{x}+\frac{y'}{y}+\frac{z'}{z}=3$: required a proof.

27. Construct the plane whose equation is just found: find its inclinations to the co-ordinate planes: the length of the perpendicular let fall upon it from the origin: the area of the tangent plane, and the volume of the pyramid formed by it and the co-ordinate planes.

28. Determine the equation to the plane touching the surface whose equation is $z(x^2+y^2)=a^3$: construct it, find its area, and the volume of the pyramid formed by it with the co-ordinate planes.

29. Find the equation to the plane touching the surface whose equation is $\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m+\left(\frac{z}{c}\right)^m=1$: construct it and determine all its circumstances.

30. If $Ax^2+By^2+Cz^2+D=0$, be the equation to a curve surface, it is required to prove that the equation of the tangent plane is $Axx'+Byy'+Czz'+D=0$: and thence to construct it, and determine all the circumstances.

31. Draw a tangent plane to the ellipsoid at the point in the plane of xz where $z=\dfrac{c}{\sqrt{2}}$, and find its inclination to the plane of xy.

32. Draw a plane parallel to a given plane, so as to touch the surface whose equation $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$.

33. Draw a tangent plane to the surface defined by the equation $ax^2-by^2+2cz=0$, and determine all its circumstances.

34. Find the equation to the plane touching the surface defined by the equation $x^2+y^2+az=b^2$, and determine all its circumstances.

35. Find the equation of the tangent plane to the surface defined by the equation

$$(az-cx)^2+(bz-cy)^2=2e\,(z-c)\,(az-cx),$$

and prove that it always passes through the same fixed point.

36. The equation of a surface being

$$(x-az)^2+(y-bz)^2=2e\,(x-az),$$

it is required to find the equation to its tangent plane, and thence to prove that it is always perpendicular to some fixed plane.

37. Draw a tangent plane to the curve surface whose equation is $a^2y-bxz+b\,(zy-cx)=e\,(a^2-z^2)$, and determine all its circumstances.

38. Determine all the circumstances of the plane which touches the curve surface defined by the equation $axy+b^2z=0$.

39. Draw a tangent plane to the surface whose equation is $(x-1)\,(y-2)\,(z-3)=(x-2)\,(y-3)\,(z-4)$, and determine all its circumstances.

40. Draw a tangent plane to the surface whose equation is $z=f(x^2+y^2)$, and find the portions it cuts off from the co-ordinate axes.

41. Apply the equation deduced in the last example to construct the plane touching at any point the surface whose equation is $(\sqrt{x^2+y^2}-c)^2=a^2-z^2$.

42. The equation to a curve surface being

$$z^2-c^2=x^2\phi\left(\frac{y}{x}\right),$$

it is required to prove that the portion of the axis of z cut off by the tangent plane at any point varies inversely as z.

43. Adapt the result of the last example to the cases of the ellipsoid and hyperboloids of one and two sheets.

44. If $bz-cy=f(bx-ay)$ be the equation of a curve surface, then will its tangent plane at every point be perpendicular to the same fixed plane: required a proof.

45. The equation of a curve surface being

$$\frac{z-c}{x-a}=\phi\left(\frac{y-b}{x-a}\right),$$

it is required to draw a tangent plane to it, and to prove that it will always pass through the same fixed point.

46. If $\log\left(\frac{z^m-a^m}{x^m}\right)=\phi\left(\frac{y}{x}\right)$, be the equation to a curve surface, it is required to prove that the plane touching it at any point cuts off from the axis of z a part proportional to $\frac{1}{z^{m-1}}$.

47. The square of the perpendicular let fall from the centre of an ellipsoid upon the tangent plane at any point, is equal to the sum of the squares of the projections of the three semi-axes upon the normal at the same point.

48. If the equation of a curve surface be

$$x^m (z + \sqrt{x^2 + y^2 + z^2}) = \phi\left(\frac{y}{x}\right),$$

it is required to prove that the tangent plane cuts off from the axis of z, a part proportional to the distance of the point of contact from the origin.

49. Find the equations of the line of greatest inclination in the planes touching the ellipsoid, the elliptic and hyperbolic paraboloids, and the hyperboloids of one and two sheets.

50. Determine the equations of the normal to any point of a sphere, and prove that it always passes through the same fixed point: find also its length.

51. Find the equation to the normal of the spheroid whose equation is $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$: prove that it always passes through the same fixed line, and determine its length and the angles which it makes with the co-ordinate planes.

52. Find the equations to the normal of an ellipsoid, and deduce the equations for drawing a normal to it which shall pass through a given point.

53. Determine the equations of the normals of the hyperbolic paraboloid, the hyperboloids of one and two sheets, the groin and the cono-cuneus of *Wallis:* and find all the circumstances of each.

54. Find the equations to the normal of the curve surface defined by the equation $(x^2 + y^2 + z^2 - a^2)^2 = 4b^2 (y^2 + z^2)$, and determine its lengths to the co-ordinate planes, and the angles it makes with them.

55. Determine the equations of the normal of the curve surface whose equation is $y = f(x^2 + y^2)$, and find all its circumstances.

56. The equation of a curve surface being

$$bz - cy = f(bx - ay),$$

find the equations to its normal, and thence deduce all its circumstances.

57. Draw a normal to the curve surface whose equation is $\frac{z-c}{x-a} = \phi\left(\frac{y-b}{x-a}\right)$, and determine all its circumstances.

58. Draw a normal to the curve surfaces included in the general equation

$$(x-a)^2 + (y-\beta)^2 + z^2 = \phi\,\{a\,(x-a) + b\,(y-\beta) + z\},$$

and point out its peculiarities.

59. Determine the angles which the normal to the curve surface whose equation is $Xdx + Ydy + Zdz = 0$, makes with the co-ordinate axes.

60. The equation of a curve surface being

$$x^2 + 2y^2 + 3z^2 = a^2,$$

it is required to find the equation to its normal, to determine its length, and to prove that it always meets the plane of xy in the point of intersection of the lines drawn from any two of the angles of the triangle AMN, to bisect the opposite sides.

61. Determine the equation to the tangent plane and also to the normal of the surface defined by the equation $z = a \tan^{-1}\left(\frac{y}{x}\right)$, and point out their peculiarities.

62. Draw a tangent plane and a normal to the curve surface whose equation is $az = \left(\sin^{-1}\frac{y}{\sqrt{x^2+y^2}}\right)^2$.

63. Draw a tangent plane and a normal to the surface whose equation is $(az+b)^2 + (bz+c)^2 = x^2 + y^2$, and determine all their circumstances.

64. Determine the tangent plane and the normal in the surface of the second order as defined by the equation

$$ax^2 + by^2 + cz^2 + \alpha x + \beta y + \gamma z = 0.$$

65. Two equal paraboloids of revolution having their axes in the same straight line, it is required to prove that the tangent plane to either makes with the other a section whose projection on a plane perpendicular to the axis, is always of the same magnitude.

66. Determine the radius of curvature of the section of an ellipsoid made by a plane passing through the axis of z, and bisecting the angle between the axes of x and y.

67. In the elliptic paraboloid it is required to determine the radius of curvature of the section made by a plane passing through the axis of z, and making an angle of 30^0 with the axis of x.

68. In a paraboloid of revolution, it is required to find the radius of curvature of any section passing through the axis of z.

69. Determine the radius of curvature of a section of the surface whose equation is $(x-a)^2 + (y-b)^2 = cz$, made by a plane passing through the axis of z: and prove that at any point the sum of the curvatures of any two sections at right angles to each other is invariable.

70. The hyperboloid of two sheets being defined by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, it is required to find the radii of curvature of its sections made by the planes of xz and yz respectively.

71. Find the normal sections of greatest and least curvature passing through the vertex of the elliptic paraboloid whose equation is $z = \frac{x^2}{a} + \frac{y^2}{b}$, and their radii of curvature at any point.

72. Determine the same in the hyperbolic paraboloid defined by the equation $abz = bx^2 - ay^2$, and shew that the sections are at right angles to each other.

73. Required the sections of greatest and least curvature through the axis of z in the cono-cuneus of *Wallis*, whose equation is $\frac{x}{\sqrt{a^2 - z^2}} + \frac{y}{b} = 1$, and determine their radii at any given point.

74. What are the sections through the axis of z, of greatest and least curvature in the surface defined by the equation $xyz = a^3$, and what are the magnitudes of their radii?

75. The equation of a surface being $z(x^2 + y^2) = a^3$, it is required to express the radius of curvature of any normal section through the axis of z, in terms of those of the sections of greatest and least curvature.

76. If the equation of a surface be $\frac{x^3}{a^3} + \frac{y^3}{b^3} + \frac{z^3}{c^3} = 1$, it is required to express the radius of curvature of any normal section through the axis of z, in terms of those of the sections of greatest and least curvature.

77. Express the radius of curvature of the section of an ellipsoid at the extremity of the axis of z made by any plane, in terms of the greatest and least radii of curvature belonging to the same point.

78. If δ_1 and δ_2 be the *minimum* and *maximum* radii of curvature at any point of an ellipsoid defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, it is required to prove that they are the roots of the equation

$$\delta^2 - \frac{a^2 + b^2 + c^2 - x^2 - y^2 - z^2}{\lambda}\,\delta + \left(\frac{abc}{\lambda^2}\right)^2 = 0,$$

where λ is the length of the perpendicular let fall from the origin upon the tangent plane.

79. It is required to shew from the last equation, that the sum of the greatest and least radii of curvature varies inversely as the length of the perpendicular upon the tangent plane, so long as the distance of the point from the centre remains invariable.

80. In the same surface, it is required to prove that so long as the distance of the tangent plane from the centre is invariable, the rectangle of the greatest and least radii of curvature is constant.

81. Determine the points in the surface of an ellipsoid at which the curvature of every normal section is of the same invariable magnitude.

82. Find the radii of greatest and least curvature at any point of the curve surface defined by the equation $xyz = a^3$.

83. Required the same for the elliptic paraboloid whose equation is $z = \frac{x^2}{a} + \frac{y^2}{b}$.

84. In any curve surface, the radii of greatest and least curvature will become equal in sign and magnitude when the conditions expressed by the equations $\frac{1+p^2}{r} = \frac{pq}{s} = \frac{1+q^2}{t}$, are fulfilled: required a proof.

85. If the radii of greatest and least curvature, having different signs, become equal in magnitude, it is required to prove that $(1+p^2)\,t + (1+q^2)\,r = 2pqs$.

86. It is required to prove that the surface whose equation is $x^2 + y^2 - (z-c)^2 = 0$, is at the same time a conical surface and a surface of revolution, by satisfying the general partial differential equation of each species of surface.

87. The equations of the curve of double curvature formed by the intersection of the surface of a sphere with that of the cylinder whose diameter is its radius, being

$$y^2 = 2ax - x^2 \text{ and } z^2 = 4a^2 - 2ax,$$

it is required to prove that the equations to the tangent line are

$$yy' + xx' = a(x + x') \text{ and } zz' = 4a^2 - a(x + x').$$

88. Shew how the line determined in the last example may be constructed, and determine where it intersects each of the co-ordinate planes, and also the loci of these intersections.

89. In the same case determine the length of the tangent intercepted between the point of contact and the plane of zy, and also its inclination to that plane.

90. The equations of the helix being

$$\frac{z}{a} = \cos^{-1}\frac{x}{b} \text{ and } \frac{z}{a} = \sin^{-1}\frac{y}{b},$$

it is required to determine the equations of its tangent line, and thence to construct it and find all its circumstances.

91. Draw a tangent line to the curve of double curvature formed by the intersection of a sphere and cylinder whose equations are $x^2 + y^2 + z^2 = a^2$ and $(x - c)^2 + y^2 = b^2$, and determine all its circumstances.

92. If the axes of two cylinders whose equations are $x^2 + z^2 = a^2$ and $y^2 + z^2 = b^2$, intersect each other at right angles, it is required to draw a tangent to the curve formed by their surfaces, and to determine all its circumstances.

93. If the axes of two right cones whose equations are $a + x = m\sqrt{y^2 + z^2}$ and $b + y = n\sqrt{x^2 + z^2}$, intersect each other at right angles, it is required to draw a tangent to the curve of double curvature thus formed, and to determine all its circumstances.

94. The equations of a curve of double curvature being $x^2 + y^2 + z^2 = a^2$ and $\tan^{-1}\frac{y}{x} = m \cos^{-1}\frac{z}{a}$, it is required to draw a tangent line to it, and to ascertain all its circumstances.

95. Draw a tangent to the spiral of *Pappus* as defined by the equations

$$\frac{z}{a} = \cos\tfrac{1}{4}\phi, \quad \frac{y}{\sqrt{a^2 - z^2}} = \sin\phi \text{ and } \frac{x}{\sqrt{a^2 - z^2}} = \cos\phi,$$

and determine all the circumstances attending it.

96. The equations to the screw of *Archimedes* being $y^2 = 2ax - x^2$ and $z = b\,\text{vers}^{-1}\frac{x}{a}$, it is required to draw a tangent line to it, and to ascertain all its circumstances.

97. It is required to draw a tangent line to the curve of double curvature defined by the equations

$$y = \sqrt{a^2 - x^2} \text{ and } \frac{z}{b} = \frac{1}{2}\left\{e^{\frac{a}{b}\sin^{-1}\frac{x}{a}} + e^{-\frac{a}{b}\sin^{-1}\frac{x}{a}}\right\},$$

and to determine all its circumstances.

98. Draw a tangent line to the curve of double curvature formed by the intersection of a right cone with the cylinder erected on the radius of the base as a diameter, and determine all its circumstances.

99. Determine the equation of the normal plane to each of the curves of double curvature mentioned in the last twelve examples and construct it:

100. Find the equation of the osculating plane in each of the curves of double curvature just referred to, and to its intersection with the normal plane.

101. In any curve of double curvature, if α, β, γ be the co-ordinates of the centre of spherical curvature, it is required to prove that they may be determined from the three equations

$$\alpha - x + (\beta -- y)\frac{dy}{dx} + (\gamma - z)\frac{dz}{dx} = 0,$$

$$(\beta - y)\frac{d^2y}{dx^2} + (\gamma - z)\frac{d^2z}{dx^2} = 1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2,$$

$$(\beta - y)\frac{d^3y}{dx^3} + (\gamma - z)\frac{d^3z}{dx^3} = 3\left\{\frac{dy}{dx}\frac{d^2y}{dx^2} + \frac{dz}{dx}\frac{d^2z}{dx^2}\right\}:$$

and the radius of the sphere from the equation

$$\delta^2 = (\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2.$$

102. If the proposed point of the curve be the origin of co-ordinates and the tangent line be considered the axis of x, it is required to prove that $\alpha = 0$,

$$\beta = \frac{\dfrac{d^3z}{dx^3}}{\dfrac{d^2y}{dx^2}\dfrac{d^3z}{dx^3} - \dfrac{d^2z}{dx^2}\dfrac{d^3y}{dx^3}}, \quad \gamma = \frac{\dfrac{d^3y}{dx^3}}{\dfrac{d^2z}{dx^2}\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2}\dfrac{d^3z}{dx^3}},$$

$$\text{and}\quad \delta^2 = \frac{\left(\dfrac{d^3y}{dx^3}\right)^2 + \left(\dfrac{d^3z}{dx^3}\right)^2}{\left(\dfrac{d^2y}{dx^2}\dfrac{d^3z}{dx^3} - \dfrac{d^2z}{dx^2}\dfrac{d^3y}{dx^3}\right)^2}.$$

103. If the absolute centre of curvature be the point where the intersection of two consecutive normal planes meets the osculating plane, it is required to determine the co-ordinates of this point, and thence the absolute curvature.

104. If the arc s of the curve be considered the principal variable, and α, β, γ the co-ordinates of the centre of absolute curvature, it is required to shew that

$$\alpha - x = \frac{\dfrac{d^2x}{ds^2}}{\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2 + \left(\dfrac{d^2z}{ds^2}\right)^2},$$

$$\beta - y = \frac{\dfrac{d^2y}{ds^2}}{\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2 + \left(\dfrac{d^2z}{ds^2}\right)^2},$$

$$\gamma - z = \frac{\dfrac{d^2z}{ds^2}}{\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2 + \left(\dfrac{d^2z}{ds^2}\right)^2},$$

$$\text{and}\quad \delta^2 = \frac{1}{\left(\dfrac{d^2x}{ds^2}\right)^2 + \left(\dfrac{d^2y}{ds^2}\right)^2 + \left(\dfrac{d^2z}{ds^2}\right)^2}.$$

105. If the point under consideration be regarded as the origin of co-ordinates and the tangent line as the axis of x, it is required to prove that

$$\alpha = 0,\quad \beta = \frac{\dfrac{d^2y}{ds^2}}{\left(\dfrac{d^2y}{ds^2}\right)^2 + \left(\dfrac{d^2z}{ds^2}\right)^2},\quad \gamma = \frac{\dfrac{d^2z}{ds^2}}{\left(\dfrac{d^2y}{ds^2}\right)^2 + \left(\dfrac{d^2z}{ds^2}\right)^2},$$

and $\delta^2 = \dfrac{1}{\left(\dfrac{d^2y}{ds^2}\right)^2 + \left(\dfrac{d^2z}{ds^2}\right)^2}$.

106. If the equation of a curve surface be of the form $u = f(x, y, z) = 0$, and $\mu^2 = \left(\dfrac{du}{dx}\right)^2 + \left(\dfrac{du}{dy}\right)^2 + \left(\dfrac{du}{dz}\right)^2$, and the notation of Article (297) be retained, it is required to prove that $\cos\alpha = \dfrac{1}{\mu}\dfrac{du}{dx}$, $\cos\beta = \dfrac{1}{\mu}\dfrac{du}{dy}$ and $\cos\gamma = \dfrac{1}{\mu}\dfrac{du}{dz}$.

107. Determine the equation to the curve traced out upon any of the co-ordinate planes by the intersection of the tangent line to a curve of double curvature whose equations are $y^2 = 2ax - x^2$ and $z^2 = m(x^2 + y^2)$.

108. Prove that the differential of the area of the surface traced out by the ordinate z in any curve of double curvature $= z\sqrt{dx^2 + dy^2}$.

109. Trace the surface whose equation is $x^2 + y^2 + z^2 = a^2$, and determine the positions of the tangent planes at the points which it cuts the co-ordinate axes.

110. Find the form of the surface defined by the equation $\left(\dfrac{x-a}{a}\right)^2 + \left(\dfrac{y-b}{b}\right)^2 + \left(\dfrac{z-c}{c}\right)^2 = 1$, determining the natures of its principal sections.

111. If an ellipsoid be cut by a plane passing through the origin perpendicular to the plane of xy, it is required to prove that the normals to the surface drawn from every point of the intersection of the plane with the ellipsoid will cut the plane of xy in a straight line, and to find the equation to that line.

112. If the partial differential equation of a curve surface be $z = a + px + qy$, it is required to shew that there is some point from which all the perpendiculars, let fall upon the tangent plane at every point, are of an invariable length.

113. The straight line joining any two points of the surface of an ellipsoid is bisected by a plane passing through the centre of the ellipsoid and through the line of intersection of the tangent planes at those points: required a proof.

114. Draw a tangent plane and a normal to the curve surface defined by the general equation $\frac{\beta x - \alpha z}{x - a} = f\left(\frac{bx - ay}{x - a}\right)$; and determine all their circumstances.

115. The equation of a surface being $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, it is required to prove that the surface of the right cone defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ will be asymptotic to it.

116. Required the asymptotic surfaces to the hyperbolic paraboloid whose equation is $z = \frac{x^2}{a} - \frac{y^2}{b}$.

117. It is required to determine the directions of the convexity and concavity of the curve surface defined by the equation $ax^2 - by^2 + 2cz = 0$, with respect to each of the co-ordinate planes.

118. The equation to a curve surface being

$$(a^2x^2 + b^2y^2 + c^2z^2)(x^2 + y^2 + z^2) - a^2(b^2 + c^2)x^2 - b^2(a^2 + c^2)y^2 - c^2(a^2 + b^2)z^2 + a^2b^2c^2 = 0,$$

it is required to determine all its circumstances; and to apply the results to the cases when $a = b$ and $a = b = c$: also when $c = 0$ and $b = c = 0$.

119. Determine the conditions necessary to be fulfilled in order that two curves of double curvature may have with each other contact of any specified order: and apply the result to the case wherein one of them becomes a straight line.

CHAP. XV.

MISCELLANEOUS EXAMPLES.

By means of the Principles laid down and exemplified in the various Articles of the Work, the solutions of the following Theorems and Problems may be effected.

1. Find the sum of the areas of all the circles that can be inscribed in an equilateral triangle touching its sides and one another, and compare it with that of the triangle.

2. If r_1 and r_2 be the radii of two spheres inscribed in a right cone, so that the greater may touch the less and also the base of the cone: it is required to prove that the content of the cone will be expressed by $\dfrac{2\pi r_1^5}{3(r_1 - r_2) r_2}$.

3. Compare the sums of the surfaces and volumes of all the spheres which can be inscribed in a right cone with those of the cone itself.

4. Differentiate from first principles the following functions:

$$u = \frac{ax}{a^2 + x^2}, \quad u = \sqrt{2ax - x^2}, \quad u = a^{mx}, \quad u = \frac{e^x}{e^x - 1},$$

$$u = \sin mx, \quad u = \frac{\cos 2x}{\sin x} \text{ and } u = \tan (2x + x^2).$$

5. If the sines of two arcs be always in a given ratio, it is required to shew that their evanescent increments are in the ratio of their tangents.

6. If the ratio of the cosines of two arcs be given, what will be the relation between their evanescent increments? Similarly, of the rest of the Trigonometrical functions?

7. If $u = \cos^{-1} x - \dfrac{2(1-x)^{\frac{1}{2}}}{1+x}$, then will $\dfrac{du}{dx} = \dfrac{\sqrt{1-x^2}}{(1+x)^2}$: required a proof.

8. If $u = \dfrac{1}{\sqrt{ab}} \tan^{-1}\left(\tan x \sqrt{\dfrac{b}{a}}\right)$, it is required to prove that
$$\frac{du}{dx} = \frac{1}{a\cos^2 x + b\sin^2 x}.$$

9. If $u = ax + b\log(a\cos x + b\sin x)$, it is required to prove that
$$\frac{du}{dx} = \frac{(a^2+b^2)\cos x}{a\cos x + b\sin x}.$$

10. If $u = \tan x \sec x + \log(\tan x + \sec x)$, it is required to shew that
$$\frac{du}{dx} = 2\sec^3 x.$$

11. Find the differential coefficients of
$$u = \log(a+x)\log(a-x) \text{ and } u = \frac{\log(a+x)}{\log(a-x)}.$$

12. Differentiate the functions $u = \log \dfrac{1+2x\sqrt{1-x^2}}{1-2x\sqrt{1-x^2}}$ and $u = \tan^{-1}\left(\dfrac{\sqrt{x}-\sqrt{1-x^2}}{\sqrt{x}+\sqrt{1-x^2}}\right)$: and find the values of the differential coefficients when $x = 0$ and $x = 1$.

13. Find the successive differential coefficients of $u = e^{e^{e^x}}$.

14. If $u = x^x$, it is required to prove that
$$\frac{d^3u}{dx^3} = -x^x\left\{\frac{1}{x^2} - \frac{3}{x}(1+\log x) - (1+\log x)^3\right\}.$$

15. From the following equations, $\sin\phi = m\sin\phi'$,
$$x = m\cos\phi' - \cos\phi \text{ and } y = m\frac{\cos\phi'}{\cos^2\phi} - \frac{1}{\cos\phi'},$$
it is required to find the values of $x + \dfrac{d^2x}{d\phi^2}$ and $y + \dfrac{d^2y}{d\phi^2}$, when $\phi = 0$.

16. If two distances u_1 and u_2 from a fixed point be such that $u_1 u_2 = a^2$, it is required to compare their evanescent increments or decrements in any position.

17. Corresponding to the extremities of the *latus rectum* of a common parabola, it is required to find the ratio of the rates of increase of the abscissa and ordinate.

18. If $u = \dfrac{a^{a-n} - x^{x-n}}{a - x}$, it is required to shew that when $x = a$, the value of $u = a^{a-n} \left\{ \dfrac{a-n}{a} + \log a \right\}$.

19. Find the value of $u = \dfrac{\sin m\theta - \cos (m \, . \, 90^0 - n\theta)}{\sin m\theta + \cos (n \, . \, 90^0 + m\theta)}$ when $\theta = 0$, m being a whole number of the form $4\mu + 1$ and n of the form $4\nu + 3$.

20. If $2y^4 - 3xy^3 + x^2y^2 - a^2x^2 = 0$, it is required to find the values of $\dfrac{dy}{dx}$ when $x = \infty$ and $y = \infty$.

21. In the equation $x^4y^4 - 2a^2x^2y^4 + a^6y^2 - a^8 = 0$, it is required to find the values of $\dfrac{dy}{dx}$ corresponding to $x = 0$ and $y = \infty$.

22. If $y = (a + bx)\, e^x + m \sin x + n \cos x$, it is required to prove that $\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$.

23. If the equation of a curve be $\dfrac{1}{y^n} = \dfrac{1}{x^n} + \dfrac{1}{a^n}$, it is required to prove that $a^n \times AT = AM^{n+1}$, where $AM = x$, $MP = y$ and PT is a tangent.

24. Describe a parabola which shall touch a circle at a given point, and have its axis coincident with a given diameter of the circle.

25. If γ and γ' be the radii of curvature of a curve and its evolute at corresponding points, it is required to prove that

$\dfrac{\gamma'}{\gamma} = \dfrac{3pq^2 - r(1+p^2)}{q^2}$, where p, q and r are the first, second and third differential coefficients belonging to the point in the curve.

26. If q, r, s and t be the second, third, fourth and fifth differential coefficients, it is required to prove that in all curves of the second order

$$9q^2t - 45qrs + 40r^3 = 0.$$

27. Apply *Lagrange's* Theorem to find a series for v in terms of nt as far as e^3 by means of the equation

$$(1-e)\tan^2\frac{v}{2} = (1+e)\tan^2\frac{u}{2}, \text{ where } nt = u - e\sin u.$$

28. If a number a be divided into such a number of parts x that the continued product of the first, the square of the second, the cube of the third, &c. shall be the greatest possible, it is required to prove that $\log x = \dfrac{x+a}{x+2a}$.

29. Into how many parts must a given number be divided, that their continued product may be the greatest possible?

30. Given the three prime factors of a number, required the indices when the number of its divisors is the greatest possible.

31. Given the three prime factors of a number, to find the indices when the sum of its divisors is the greatest possible.

32. If m be an even number, it is required to prove that

$$\frac{m}{1-x^m} = \frac{2}{1-x^2} + \frac{2 - 2x\cos\dfrac{2\pi}{m}}{1 - 2x\cos\dfrac{2\pi}{m} + x^2} + \frac{2 - 2x\cos\dfrac{4\pi}{m}}{1 - 2x\cos\dfrac{4\pi}{m} + x^2} + \&c.$$

to $\frac{1}{2}m$ terms. See Article (346).

33. Decompose $\dfrac{1}{x^m+1}$ into a series of fractions as in the last example, both when m is odd and even.

34. If f denote any function whatever, and we have

$$f(x+h)=f(x)\,f(h),$$

it is required to prove that $f(h)=e^{ah}$. See Article (348).

35. Find the form of the function f so that we shall have $f(x+h)-f(x-h)=f(x)\,f(h)$.

36. Required the form of the function f such that

$$f(x-h)=\frac{f(x)}{f(h)}.$$

37. Required the value of $u=\dfrac{(x^2+a^2)\,(x^2-a^2)^{\frac{3}{2}}}{(3x^3-a^2x)\,(x^3-a^3)^{\frac{3}{2}}}$ when $x=a$, according to the principle laid down in Article (349).

38. Determine the arc of a given circle, when the rectangle of its sine and the excess of its sine above the cosine is a *maximum*, and verify the result.

39. In Article (351) it is required to find the straight line which drawn through P, shall cut off the segments BT and BV, so that their sum shall be the least possible.

40. Retaining the same figure, find the position of the line TV so that the area of the triangle TBV shall be the least possible.

41. Of all triangles under a given perimeter and one determinate side, prove that to be the greatest in which the two indeterminate sides are equal.

42. Of all triangles upon the same base and of a given area, it is required to prove that the isosceles triangle has the greatest vertical angle.

43. Find the greatest of all triangles having equal vertical angles and equal distances between that angle and the bisection of the opposite side.

44. Of all equiangular parallelograms inscribed in a given triangle, the greatest is that formed by the bisection of the sides: required a proof.

45. Given the difference of the angles at the base and the radius of the inscribed circle, to find the triangle whose perimeter is a *maximum*.

46. Find the area of the greatest rectangle that can be inscribed in a given triangle, so that one of its sides is parallel to the base of the triangle.

47. A circle and an ellipse having the same major axis, it is required to compare the areas of the greatest rectangles that can be inscribed in them.

48. It is required to prove that of all triangles described on the same base and having equal altitudes or vertical angles, the perimeter of the isosceles is the least.

49. Of all triangles upon the same base and of equal perimeters, it is required to find that which has the greatest area.

50. In a given line, it is required to determine the position of a point at which another given line subtends the greatest possible angle.

51. Find the greatest rectangle that can be inscribed in a given segment of a circle.

52. Draw a tangent to an ellipse so that the part of it intercepted between the axes produced shall be a *minimum*.

53. Find the shortest line which can be drawn touching a given ellipse and intercepted by the tangents at the extremities of its axes.

54. Prove that the perimeter of an equilateral triangle inscribed in a circle is greater than the perimeter of any other isosceles triangle inscribed in the same circle.

55. Find the greatest isosceles triangle that can be inscribed in a given circle.

56. Determine the points of a given ellipse in which the sum of the conjugate diameters is the greatest or least possible, and distinguish the *maximum* from the *minimum*.

57. Determine those conjugate diameters in an ellipse or hyperbola whose rectangle is the greatest possible.

58. In a given ellipse, it is required to assign the positions of those conjugate diameters which include the greatest and least angles.

59. It is required to determine the magnitude of the greatest parabola that can be made by cutting a given cone.

60. Inscribe in a portion of a parabola cut off by a double ordinate the greatest possible rectangle.

61. In a given parabola, inscribe another parabola whose area shall be the greatest possible, the vertex of the inscribed parabola being any given point in the axis of the other.

62. Of all the straight lines which can be drawn from the vertex of a given ellipse to the circumference of its circumscribed circle, find that of which the portion intercepted between the circumferences of the ellipse and circle shall be the greatest possible.

63. Two circles of given radii intersect each other, find the longest straight line which can be drawn through either point of intersection, and terminated by the circumferences.

64. In an ellipse, it is required to find the point at which the normal makes the greatest angle with the line drawn from it to the centre.

65. If M be any point in the diameter of a circle whose centre is C, PMQ a chord and CP be joined; prove that the chord PQ is a *minimum* and the angle CPQ a *maximum*, when PQ is perpendicular to the diameter.

66. Perpendiculars Aa, Bb are drawn from the extremities A and B of the axis major of a given ellipse, and through a given point P in its circumference is drawn a straight line aPb intersecting them in a and b respectively: prove that the rectangle $Aa \cdot Bb$ is a *maximum* when aPb is a tangent to the ellipse at P.

67. Two given points A and B are situated on the same side of a given straight line: find a point P in the line such that $AP+BP$ shall be a *minimum*. Also, if the straight line move parallel to itself, determine the locus of the point P.

68. Given the length of a chord drawn through the vertex of a common parabola; find its *latus rectum* when the corresponding area APM is the greatest possible.

69. Find the point in the diameter of a given circle, through which an ordinate being drawn, the m^{th} power of the ordinate multiplied by the n^{th} power of the less abscissa shall be a *maximum*.

70. In a given ellipse it is required to find a point from which straight lines drawn to the extremities of the axes shall contain the greatest angles.

71. If P and Q be corresponding points in an ellipse and the circle described upon its major axis, it is required to prove that the angle contained between the tangents at P and Q will be a *maximum* when $AM=a\left\{1-\sqrt{\dfrac{a}{a+b}}\right\}$.

72. For the same point P it is required to prove that

$$CM=a\sqrt{\frac{a}{a+b}},\quad MP=b\sqrt{\frac{b}{a+b}},$$

$$CP=\sqrt{\frac{a^3+b^3}{a+b}},\quad MT=b\sqrt{\frac{a}{a+b}},\quad TP=b:$$

and if CF be drawn perpendicular to the tangent at P,

$$CF=\sqrt{ab},\quad TF=a \text{ and } PF=a-b \text{ a } maximum.$$

73. Prove that the area of the triangle formed by the tangent and the axes produced is a *minimum* corresponding to the co-ordinates $a\dfrac{\sqrt{2}-1}{\sqrt{2}}$ and $b\sqrt{\dfrac{2\sqrt{2}-1}{2}}$.

74. Through a given point in the diameter of a given circle it is required to draw a chord which shall form with the

lines joining its extremities with either extremity of the diameter, the greatest possible triangle.

75. Through a given point in the surface of a right cone it is required to pass a plane, so that the area of the elliptic section shall be the least possible, and to find the limitations of the problem.

76. Determine the greatest ellipse that can be formed by cutting a right cone, and find the limitations to which the problem is subject.

77. From the extremity of the minor axis of an ellipse it is required to draw a line such that the part of it cut off by the periphery may be the longest possible, and to determine the limitations of the problem.

78. In a given straight line it is required to find the equation which determines a point P, such that straight lines PA and PB being drawn to the given points A and B on opposite sides of it, the sum of $m \times AP$ and $n \times BP$ shall be a *minimum:* and to compare the angles made by AP and BP with the proposed line.

79. Two points are taken in a wall and joined by a string which passes about a corner of it, prove that this string is the shortest possible when the parts of it make equal angles with the line which forms the corner of the wall.

80. A circle being given and a point without it, it is required to find the circle of which this point is the centre when the segment of it inscribed in the first circle is a *maximum*.

81. Determine the point in a curve whose equation is $a^{m-1}x = y^m$, to which a line must be drawn from the vertex making the greatest angle with the curve.

82. If the semi-axes of an ellipse be $2x^x$ and x^x, it is required to find the least area the ellipse admits of.

83. The corner of the leaf of a book being doubled back so as just to reach the other edge of the page, it is required to find where the length of the crease will be the least possible.

84. Inscribe the greatest trapezium having two of its sides parallel in a given parabola, and compare its area with that of the parabola.

85. Find the radius of the circle, such that the versed sine of an arc being given, the arc itself may be a *minimum*.

86. Two points move at the same time from two given positions at given rates in given directions, find where their distance from each other will be the least possible.

87. Required the triangle of a given perimeter which has the greatest possible area.

88. Find the greatest triangle that can be inscribed in a circle of given radius.

89. The three angles of a right-angled triangle are at given distances from a fixed point, find the greatest area the triangle can admit of.

90. It is required to describe the greatest quadrilateral figure in a circle, and the least quadrilateral about it.

91. Given one of the angles and the perimeter of a plane triangle, to find the sides when the area is the greatest possible.

92. Given the three sides of a triangle, to find a point such that the continued product of the three perpendiculars let fall from it upon the sides may be a *maximum*.

93. Given the four sides of a trapezium taken in order, it is required to find its area when the sum of the squares of the diagonals is a *maximum*.

94. In a given triangle it is required to determine the position of a point from which if perpendiculars be drawn to the sides, the sum of their squares shall be a *minimum*.

95. To find a point within a given triangle, from which if lines be drawn to the angular points, the sum of their squares shall be the least possible.

96. Find a point within a given trapezium from which perpendiculars being let fall upon the sides, their sum shall be of a given magnitude and their continued product a *maximum*.

97. Find a point within a given quadrilateral from which lines being drawn to the angular points, the sum of their squares shall be the least possible.

98. Required the position of the point in the line joining the centres of two given spheres, from which the sum of the spherical surfaces visible shall be a *maximum*.

99. Required the same when the sum of the corresponding spherical segments is the greatest possible.

100. Among all the angles contained in a given parabolic segment, determine that which is the greatest.

101. Corresponding to a spherical surface of given magnitude, it is required to prove that a hemisphere is greater than any other segment of a sphere.

102. Let it be required to inscribe the greatest cylinders in a right cone, a sphere and a paraboloid: and to compare their volumes in each case.

103. Inscribe the greatest cones in a spheroid, and a paraboloid, the axes and vertices of the solids being coincident.

104. Of all cones inscribed in a given sphere, determine that which has the greatest convex surface.

105. In a given segment of a sphere, inscribe a paraboloid so that the content shall be a *maximum*, the vertex of the paraboloid coinciding with the centre of the base of the segment.

106. Find the dimensions of the greatest cylinder that can be cut out of a solid formed by the revolution of a curve about its axis, of which the equation is $a^m x^n = y^{m+n}$, and the length of the axis $= b$.

107. The volume of a cylinder being given, it is required to find whether the convex and entire surfaces admit of a *maximum* or a *minimum*.

108. A cylindrical vessel of given thickness is required to be of a given capacity, find the least quantity of material whereof it can be made.

109. Describe the greatest parallelopiped in a given sphere, and the least parallelopiped about it; and compare their volumes and surfaces.

110. The surface of a right cone being given, find its dimensions when the volume is the greatest possible.

111. Given the volume of a right cone, required its dimensions when the whole surface is the least possible.

112. Determine the rectangular parallelopiped of a given volume, which has the least possible surface.

113. A certain quantity of material is to be formed into a spherical segment of given thickness, what will be its dimensions when the included volume is a *maximum?*

114. A given sphere is to be formed into a solid composed of two equal right cones on opposite sides of a common base, so that the surface is a *minimum:* find the dimensions of the cones and compare their surfaces with that of the sphere.

115. In a right-angled spherical triangle is given an angle to find the sides, when the difference between the hypothenuse and the side adjacent to the given angle is a *maximum.*

116. If two sides of a spherical triangle be given, it is required to find when the area will be the greatest possible.

117. If the hypothenuse of a right-angled spherical triangle be given, find its dimensions when the sum of the cosines of the sides is the least possible.

118. If the sum of two sides, and the included angle of a spherical triangle be given, it is required to determine when the sum of the remaining angles will be the least possible.

119. Determine that point in the arc of a quadrant from which two lines being drawn, one to the centre and the other bisecting the radius, the included angle shall be the greatest possible.

120. Of all isoperimetrical polygons having the same number of sides, the greatest is that which is equilateral: required a proof.

121. Describe the least paraboloid about a given cylinder, when their axes are coincident in direction.

122. Find that point in the surface of a given paraboloid through which if two planes be drawn, one perpendicular to the axis and the other perpendicular to the generating parabola and parallel to the axis, the sum of the areas of the sections shall be the greatest possible.

123. On a given triangle a pyramid is to be constructed of a given volume; required its dimensions when the surface is the least possible.

124. Find that point in the surface of a spherical triangle, from which if straight lines be drawn to the angular points the pyramid thus formed shall be the greatest possible.

125. Given the greatest and least slant sides of an oblique cone to find the diameter of the base, when the volume is the greatest possible.

126. Required the area of the least isosceles triangle that can circumscribe a given circle, and compare their perimeters.

127. Find the content of the least cone that can circumscribe a given sphere, and compare their surfaces.

128. It is required to inscribe the greatest parabola in a given isosceles triangle, the axis of the parabola coinciding in direction with the line bisecting the vertical angle of the triangle.

129. Find the area of the least triangle which can circumscribe a given segment of a parabola.

130. Determine the content of the greatest paraboloid that can be inscribed in a right cone of given dimensions.

131. Find the content of the least cone that can circumscribe a paraboloid of given dimensions.

132. Determine the dimensions of the least paraboloid that can circumscribe a given sphere.

133. Required the least triangle which can circumscribe any given right segment of an ellipse.

134. Determine the dimensions of the greatest spheroid and hemispheroid that can be inscribed in a given right cone.

135. Determine the least triangle that can circumscribe any segment of an ellipse.

136. Find the content of the least cone that can be circumscribed about any right segment of a spheroid.

137. Find the least triangle that can be circumscribed about an ellipse of given dimensions.

138. Required the dimensions of the least ellipse that can be circumscribed about a given trapezoid.

139. Inscribe an ellipse in a given parallelogram so that its area may be the greatest possible.

140. Inscribe the greatest ellipse in a given segment of a circle, one axis of the ellipse being parallel to the base of the segment.

141. In a given ellipse inscribe the greatest semi-ellipse, the extremity of the major axis of the inscribed figure touching the extremity of the axis minor of the given ellipse.

142. Inscribe the greatest spheroid in a given paraboloid, and compare their axes and volumes.

143. If a curve be defined by the equation $a^m x^n = b^n y^m$, where x and y are the distances of any point of it from two given points, it is required to draw a tangent and a normal to it.

144. If $m = n = 1$, and the curve become a circle, it is required to determine the position of the tangent, and to shew that the normal always passes through the same point.

145. If the arc of a cycloid be considered a curvilineal abscissa and the corresponding ordinate be always equal to it, it is required to draw a tangent to the curve, and to find where it meets the co-ordinate axes.

146. It is required to draw a tangent to two given circles whose centres are at a given distance from each other.

147. An ellipse and a circle given in magnitude and position have their axes in the same straight line, it is required to draw a straight line which shall touch them both.

148. A given ellipse and parabola have their principal axes in the same straight line, it is required to draw to them a common tangent.

149. If perpendiculars be drawn from any given point in the circumference of a circle upon the tangents, it is required to find the locus of their intersections.

150. It is required to find the nature of the line traced out by the intersections of the tangents to a parabola with the perpendiculars let fall upon them from the focus.

151. Determine the locus of the intersections of the perpendiculars from the focus of an ellipse with the tangents at every point of it.

152. Determine the locus of the intersection of the tangent to an hyperbola with the perpendicular let fall upon it from the centre.

153. Find the equation to the curve traced out by the intersections of the tangents to the conic sections with the perpendiculars let fall upon them from any given point.

154. In each of the conic sections a straight line being drawn from the focus to any point in the curve, it is required to find the locus of the intersection of a perpendicular to this line through the focus with the tangent.

155. Find the locus of the point from which two tangents drawn to a circle given in magnitude and position, shall always include the same given angle.

156. Find the locus of the centre of a circle touching a given line and always passing through the same fixed point.

157. Determine the locus of the centre of a circle which shall touch a given straight line and a given circle: also when it touches two given circles.

158. Required the locus of the intersection of two tangents to a given parabola, when the angle included between them is invariable.

159 Required the locus of the intersection of two tangents to a parabola inclined to its axis at angles, the product of whose trigonometrical tangents is invariable.

160. Determine the locus of a point from which two tangents being drawn to a given parabola, their sum or difference shall be invariable.

161. Find the locus of a point from which two tangents being drawn to a given parabola, their lengths shall have an invariable ratio.

162. Required the locus of the point from which the rectangle of two tangents, drawn to a given parabola, shall be of invariable magnitude.

163. Determine the locus of the intersections of the tangents to an ellipse or hyperbola, which shall be inclined to the principal axes at angles, the products of whose trigonometrical tangents are invariable.

164. Required the nature of the loci, when the tangents to the ellipse or hyperbola intersect each other at right angles.

165. Find the locus of the intersections of two tangents to an ellipse or hyperbola, when the sum of the angles made by them with the principal axes, is equal to a right angle.

166. Required the loci of the intersections of the tangents to an ellipse or hyperbola, which are parallel to conjugate diameters.

167. Determine the locus of a point from which tangents drawn to a given ellipse, shall have an assigned ratio.

168. Required the locus of a point, from which tangents being drawn to a given ellipse, they shall have an assigned sum or difference.

169. Required the nature of the line which touches all equal circles having their centres in the same given straight line.

170. Determine the nature of the line which touches all circles passing through a given point, and having their centres in the same given straight line.

171. Of a right-angled triangle, if the sum of the sides including the right angle be the given magnitude a, it is required to prove that the curve to which the hypothenuse is always a tangent, will be defined by the equation

$$\sqrt{x} + \sqrt{y} = \sqrt{a}.$$

172. Find the nature of the curve to which the hypothenuse of a right-angled triangle is always a tangent, when the difference of the sides including the right angle is invariable.

173. Required the nature of the curve which a straight line of given length having its extremities always in two straight lines at right angles to each other, continually touches.

174. If PT be a tangent to the curve AP, and AD be drawn perpendicular to the axis of x through the origin, it is required to find the nature of the curve wherein $AT \propto AD^m$.

175. Required the nature of the curve formed by the intersections of consecutive normals to all the points of a given parabola.

176. Prove that the curve which the normals to every point of a given cycloid perpetually touch is a similar and equal cycloid in any inverted position.

177. If a straight line be given in position, and lines be drawn intersecting the given line so that the perpendiculars to them from the points of intersection shall pass through a given point, find the curve to which these lines are always tangents.

178. In two straight lines at right angles to each other, two points are given in position, and two other points at equal distances from them are taken: it is required to find the nature of the curve to which the straight line joining the latter points is always a tangent.

179. If normals and tangents to a parabola be always drawn at the extremities of any parameter, it is required to find the loci of their intersections.

180. Determine the nature of the curve which shall include all isosceles triangles of a given area, having their bases and perpendiculars coincident in directions.

181. Two straight lines are drawn through two given points whose equations are $x = m$ and $x = n$, perpendicular to the axis of x, and their rectangle is of invariable magnitude: it is required to find the curve to which the line joining their extremities is always a tangent, both when they are drawn in the same and in opposite directions.

182. From two given points two straight lines are drawn parallel to each other so that the difference of their squares is of invariable magnitude, it is required to find the nature of the curve to which the straight line joining their extremities is always a tangent.

183. Required the nature of the curve analogous to that of the last example, when the sum of the squares of the lines cut off by the tangent is invariable.

184. Two points are assumed in two given lines and a straight line intersects them so that the rectangle of the distances of its intersections from the said points is invariable: required the curve to which the straight line is always a tangent.

185. Three points A, B and C being given in position, and AB, AC being drawn, a straight line TV meeting these lines is so situated that the rectangle $BT \times CV$ has to the rectangle $AV \times AT$ an invariable ratio: required the nature of the curve to which this line is always a tangent.

186. Required the nature of the curve which shall cut all curves of a given species, so that the tangents at the points of intersection may pass through the same given point.

187. Apply the result of the last example when the curves of given species are a series of parabolas defined by the equation $y^2 = ax$.

188. Required the curve when the curves of given species are a series of ellipses, having one of their principal axes invariable.

189. Find the nature of the curve when the curves of given species are a series of ellipses, having their principal axes in a given ratio.

190. Determine the nature of the curve when the curves of given species are cissoids of *Diocles*, having their axes coincident in direction.

191. Find the nature of a curve which shall cut all curves of a given species, so that the tangents at the points of intersection may be of an invariable length: and apply the result when the curves proposed are a series of parabolas.

192. Determine the nature of the curve cutting all curves of a given species, so that the tangents at the points of intersection may touch a given circle.

193. Determine the nature of the curve which shall bound a series of circles, whose respective radii are the successive ordinates of a common parabola.

194. Find the nature of the curve which shall touch all parabolas whose areas are equal, and ~~which pass through a given point and~~ have their extreme ordinates coincident in direction.

195. If the intersection of two straight lines, which contain a given angle, be upon a line of any kind given in position, and one of them pass through a given point, required the nature of the curve to which the other is a tangent.

196. Apply the result of the last example to the cases, wherein the line of the proposed kind becomes a straight line and a circle.

197. Let AEC, BFC and EF be three straight lines given in position, and from the points A, B let straight lines be drawn to any point in the straight line EF meeting AC in P and BC in Q: then it is required to find the nature of the curve to which the straight line joining P and Q is always a tangent.

198. Prove that all the straight lines defined by the equation $y - cx = a\sqrt{1 + c^2}$ by assigning different values to c, are tangents to the circle whose equation is $y = \sqrt{a^2 - x^2}$.

199. Determine the nature of a curve when its tangent is defined by the equation $(c - 1)\, x - cy = a - b$, whatever be the value of c.

200. Find the curve to which the straight line belonging to the equation $y = c\,(x - a) + b\sqrt{1 + c^2}$ is always a tangent, whatever be the value of c.

201. Required the curve to which the straight lines whose equations are $y = c\,(a - x) \pm \sqrt{b^2 + c^2 a^2}$ are always tangents, whatever value be assigned to c.

202. If the equation to a straight line be $\frac{x}{\alpha} + \frac{y}{\beta} = 1$, where α and β are subject to the equation $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{1}{c^2}$, it is required to prove that the curve formed by the intersections of all such lines will be a circle whose equation is $x^2 + y^2 = c^2$.

203. If an infinite number of straight lines defined by the equation $\alpha y + \beta x = \alpha\beta$ be drawn, subject to the condition expressed by the equation $\alpha^n + \beta^n = c^n$, then will the curve whose equation is $x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}}$, touch them all.

204. Determine the nature of the curve which shall bound all the parabolas expressed by the equation $y = \alpha x - (1 + \alpha^2)\, x^2$, whatever value be assigned to α.

205. Find the nature of the line which shall touch the curve defined by the equation $y^2 = ax - a^2$, whatever be the value of a.

206. If any number of concentric ellipses have their axes coincident in direction and their sum $= 2a$, it is required to shew that the curve whose equation is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, will bound them all.

207. If the sum of the squares of the axes of any number of concentric ellipses, having their axes coincident in direction, be invariable, it is required to prove that a straight line will bound them all.

208. The equation to a series of ellipses being

$$\left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 = 1,$$

subject to the condition expressed by the equation $\alpha^n + \beta^n = c^n$, it is required to find the equation to the curve which touches them all.

209. Required the nature of the curve which shall touch all the curves defined by the equation $(x^2 + y^2)^2 = \alpha^2 x^2 + \beta^2 y^2$, wherein $\alpha\beta$ is invariable.

210. Determine the nature of the curve which shall touch all the curves defined by the equation $y = ax - (a + 4)\, ax^2$, whatever be the value of a.

211. Given the focus of a parabola and a point through which it passes, to determine the locus of the vertex.

212. Given the focus of a parabola and the position of a tangent, to determine the locus of the vertex.

213. Find the locus of the foci of all parabolas having a given vertex and touching a given straight line.

214. Given one focus of an ellipse, a point through which it passes and the position of a tangent at that point, to determine the loci of its vertices.

215. The centre of an ellipse and its axis major being given, it is required to find the loci of the extremities of the

minor axis, when the ellipse always passes through a fixed point.

216. Given one focus of an ellipse, a point through which it passes and the magnitude of the major axis, to determine the locus of the vertex.

217. Given the centre of an ellipse, a point through which it passes, and the position of a tangent at that point, to find the locus of the focus.

218. Ellipses having a given centre and eccentricity touch a given line, it is required to find the loci of the extremities of their major axes.

219. Given the position of the centre of an hyperbola and the magnitude of its major axis, to find the locus of the focus when it always passes through a given point.

220. Given the position of the vertex of an hyperbola, a point through which it passes and the position of a tangent at that point, to determine the locus of the focus.

221. Given the vertex of an hyperbola, a point through which it passes and the magnitude of the major axis, to determine the locus of the focus.

222. Find the locus of the centre of an ellipse which touches two given straight lines at two given points.

223. If two sides of a triangle be divided into the same number of equal parts and the points of division of one side beginning from the base be joined with those of the other side beginning from the vertex, find the curve in which the intersections of every pair of consecutive lines taken in this order are situated.

224. If on two sides of an isosceles triangle equal parts be assumed, it is required to find the curve in which the points of intersection of every pair of lines joining the successive points of division are situated.

225. Determine the locus of the vertices of the osculating parabolas to a given curve; and prove that for the cubical parabola, it is a curve of the same species.

226. If tangents be drawn from a given point to each of a given system of curves: shew generally how to determine the curve which is the locus of all the points of contact: and apply the method when tangents are drawn from a given point to a system of concentric similar ellipses.

227. By means of the equation to the tractrix, it is required to determine the length of the catenary.

228. Determine the length of any portion of the curve whose equation is $(y+x)^{\frac{2}{3}}-(y-x)^{\frac{2}{3}}=a^{\frac{2}{3}}$, by means of the rectangular hyperbola referred to its asymptotes.

229. Required the length of that part of the evolute of an hyperbola which is included within the branches of the hyperbola, and determine the angle in which the two curves intersect each other.

230. From the equation between polar co-ordinates, it is required to prove that the locus of the intersections of the tangents to a parabola with perpendiculars upon them from the focus is a straight line.

231. Find the equation to the locus of the intersection of the tangent to a circle with the perpendicular let fall upon it from any point in the circumference.

232. Required the same when perpendiculars are drawn upon the tangents of a rectangular hyperbola from the centre.

233. In the involute of a circle where the given point is the centre, it is required to prove that the corresponding locus is the spiral of *Archimedes*.

234. It is required to prove that in each of the conic sections, the locus of the extremity of the polar subtangent from the focus is a straight line.

235. Shew that the extremity of the polar subtangent of the logarithmic spiral traces out the involute, and that of the polar subnormal the evolute, of that curve.

236. Determine the curve from every point of which perpendiculars drawn to a given parabola are equal.

237. If several chords be drawn through the focus of a conic section, and through the points where they intersect the curve, tangents be drawn, it is required to prove that the locus of the corresponding intersections is a straight line.

238. Determine the curve from every point of which perpendiculars drawn to a given ellipse or hyperbola are equal.

239. A straight line being drawn from the centre of an ellipse to meet the tangent and parallel to the distance from the focus, it is required to find the locus of their intersections.

240. Inscribe an ellipse in a given triangle so as to touch two sides in given points.

241. Inscribe in a triangle an ellipse of given area, so as to touch one side in its middle point.

242. Among the several parabolas which may circumscribe a given triangle, to determine that whose segment cut off by any side of the triangle, shall be the least possible.

243. In an ellipse or hyperbola, it is required to determine the locus of the extremity of the polar subtangent, the centre being assumed as the pole.

244. A right line being given in position, it is required to find the points in it, from which two tangents drawn to a given parabola, shall contain the greatest or least possible angle.

245. Find the equation of an ellipse touching the three sides of a given triangle, and shew how far the curve is indeterminate.

246. In the circumference of the circle described on the major axis of an ellipse, it is required to determine that point from which two tangents to the ellipse shall include the least possible angle.

247. Describe the least parallelogram about a given ellipse, and compare the areas of the two figures.

248. One circle rolls upon another equal to it, it is required to determine the locus of any point of its circumference.

249. Of two ellipses having at first their vertices coincident and their axes in the same directions, one rolls upon the other, it is required to find the loci of the vertex and focus of the moveable ellipse.

250. From the vertex of a parabola a straight line is drawn inclined at an angle of 45^0 to the tangent at every point: find the equation to the curve which is the locus of their intersections.

251. To determine the greatest ellipse inscribable in a given triangle touching one of the sides at a given point.

252. Prove that the line, joining the points of contact of rectangular tangents to the cardioide, passes through a fixed point.

253. Determine the locus of the intersections of rectangular tangents drawn to the cardioide.

254. If a circle be inscribed in an ellipse and a common ordinate to the minor axis be drawn, it is required to prove that the locus of the intersections of the normals at the corresponding points of the ellipse and circle is a concentric circle whose radius is equal to the sum of the semiaxes.

255. If about a given point D the angle BDC revolve, cutting the straight lines AB and AC given in position in the points B and C: it is required to prove that the curve which is always touched by the straight line BC is a conic section.

256. If tangents be drawn making given angles with the axes of all conic sections having the same given foci, it is required to find the curve which is the locus of the points of contact.

257. If α and β be two sides of a triangle subject to the condition expressed by the equation $\alpha^2 + \alpha\beta = c^2$, it is required to find the nature of the curve to which the third side

is always a tangent, when the angle included between α and β is of invariable magnitude.

258. Lines being drawn to cut the three sides of a triangle, so that the continued product of the parts estimated from the angles is invariable; it is required to find the curves to which these lines are always tangents.

259. A series of harmonic curves with the same co-ordinate axes and equal maximum ordinates, have tangents drawn to them at the points where they meet the axis: it is required to find the curve which is formed by the continual intersections of these tangents.

260. In a given parabola, it is required to find the locus of a point which shall always divide the normal into two parts having a given ratio.

261. Determine when one proposed function is a function of another involving the same principal variables.

262. Of all sections of a paraboloid of revolution made by planes passing through a tangent at its base, it is required to find that whose area is the greatest.

263. Inscribe in a portion of the elliptic or hyperbolic paraboloid cut off by a plane parallel to that of xy, the greatest parallelopiped.

264. Draw a plane through a given point equidistant from three co-ordinate planes at right angles to one another, forming with them a pyramid the area of whose three faces shall be the least possible.

265. A given square is placed in a spherical surface and four planes are drawn through the centre of the sphere and the sides of the square: determine the surface cut off.

266. If x and y be the co-ordinates to any point P in the plane curve APQ, $x+h$ the abscissa belonging to any other point Q in the curve and $AP=s$; prove that the surface of the solid generated by the revolution of the arc PQ about the axis of x, is equal to

$$2\pi h\left\{u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\&c.\right\},\text{ where } u=\frac{yds}{dx}.$$

267. Draw a plane which shall touch two spheres given in position and magnitude, and shall pass through a given point.

268. Find the equation of the curve surface traced out by one of the angles of a square in constant contact with a spherical surface, without sliding upon the surface.

269. If three tangent planes to an ellipsoid be mutually at right angles, it is required to prove that their point of intersection will trace out a concentric spherical surface.

270. There is an indefinite number of spherical caps, having equal surfaces but different radii and a common tangent plane at their poles: it is required to determine the surface in which the circumferences of all their bases are situated.

271. What surface will be always touched by a series of planes so drawn that the rectangles of the perpendiculars let fall upon them from two fixed points shall be invariable?

272. If two concentric surfaces of the second order, have the same foci for their principal sections, it is required to prove that they will every where intersect each other at right angles.

273. If three planes mutually at right angles to each other, touch a plane curve of the second order, it is required to prove that the locus of their intersections is the surface of a concentric sphere.

274. Required the nature of the surface touching a series of planes which cut off from a given right cone, a portion towards the vertex whose volume is invariable. Similarly of a parabola of revolution.

275. Determine the nature of the surface which shall envelope all equal paraboloids of revolution passing through a given point and having their axes parallel to a given line.

276. If the equation of a surface be

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{\beta}\right)^m + \left(\frac{z}{\gamma}\right)^m = 1,$$

subject to the condition expressed by the equation

$$a^n + \beta^n + \gamma^n = c^n:$$

required the equation to the surface which shall touch all surfaces possessing this property.

277. Required the nature of the surface touching all such surfaces when the equation of condition is $a\beta\gamma = c^3$.

278. If an oblate spheroid revolve about any one of its diameters, it is required to determine the nature of the surface which shall envelope it.

279. If all possible ellipsoids be described of which the semiaxes a, β, γ, are coincident in directions, and subject to the conditions expressed by $\frac{a}{\beta} = \frac{\gamma}{a} = \frac{\beta}{c}$: it is required to determine the surface which shall touch them all: and to find the equations of the curve in which it is touched by any one of the ellipsoids, and of the curve which is the locus of the intersections of all such curves.

280. Required the greatest possible equilateral triangle, which shall have its three sides passing through three points given in position.

281. From two points given in position, it is required to draw to a point in a given plane, two straight lines whose sum shall be the least possible.

282. Of a right cylinder there is given the length of a line drawn from the circumference of one of the ends to the centre of the other: it is required to find the dimensions, when its volume is the greatest possible.

283. Determine the position of a luminous point, in the plane passing through the centres of three given spheres, so that the sum of their illuminated surfaces may be the greatest possible.

284. Required the shortest straight line which can be drawn through a given point, and terminated by two straight lines whose equations are given.